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MATHEMATICS MAGAZINE

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VOLUME 44 · NOVEMBER 1971 · NUMBER 5

MATHEMATICS MAGAZINE

GERHARD N. WOLLAN, *EDITOR*

EDITORIAL CORRESPONDENCE should be sent to the EDITOR, G. N. WOLLAN, Department of Mathematics, Purdue University, Lafayette, Indiana 47907. Articles should be typewritten and triple-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

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ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Washington, D. C., bi-monthly except July–August. Ordinary subscriptions are: 1 year \$4.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of 2 years for \$6.00. Single copies are 80¢.

Second class postage paid at Washington, D. C. and additional mailing offices.

NOTES AND COMMENTS

E. R. Seelbach and F. E. Watkins of the University of Wyoming note that the main theorem, Proposition 1, of the article *On properties preserved by continuous functions* by Michael Gemignani, pp. 181–183, September 1968, is false. They give a simple example of a regressive property which is preserved by any one-one map but is not preserved by some maps.

Albert Wilansky, Lehigh University, comments on *The distribution of quadratic residues in fields of order* p^2 by Hardman and Jordan, pp. 12–17, January 1969, as follows: "The results can be generalized to arbitrary odd primes and with an elementary proof. Let p be an odd prime, z a quadratic nonresidue, F the field $\{a+bu\colon a, b \text{ integers} \pmod p,\ u^2=z\}$. For $z=-1$ we obtain the special case of the article.

"THEOREM 1. *u is a square in F if and only if $-z/4$ is congruent* (mod p) *to a fourth power.*

"THEOREM 2. $\langle(a+bu)/p\rangle=((a^2-zb^2)/p)$.

"Here (t/p) is the Legendre symbol. If $a+bu=(c+du)^2$ then $a^2-zb^2=(c^2-zd^2)^2$. Conversely if $a^2-zb^2=e^2$, either $(a\pm e)2z$ is a square since their product is a nonsquare. Write $(a+e)/2z=f^2$. Then $[b/2f+fu]^2=a+bu$."

R. P. Boas writes that there is a mistake (not a serious one) in Reed's note in the November 1969 MAGAZINE. He applies Lhospital's rule to $(\int_1^n f(x)dx)/g(n)$, but his hypotheses don't imply that this is actually an indeterminate form.

Lloyd G. Roeling of the University of Southwestern Louisiana comments on (1) S. K. Hildebrand and H. W. Milnes, *An interesting metric space*, pp. 244–247, November 1968, and (2) R. Shantaram, *On an interesting metric space*, pp. 95–97, March 1970. He notes that if for $x=(x_1, \cdots, x_n)$, $y=(y_1, \cdots, y_n)$ in R^n we define $\eta_n(x, y) = \sum_{i=1}^n |x_i-y_i|$ and

$$\phi_n(x, y) = \begin{cases} \|x - y\| & \text{if } x=ry \text{ for some } r \text{ in } R, \\ \|x\| + \|y\| & \text{otherwise,} \end{cases}$$

where
$$\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{1/2},$$

then $\eta_n(x, y)$ is a metric on R^n which is translation invariant but not rotation invariant and $\phi_n(x, y)$ is a metric on R^n which is rotation invariant but not translation invariant. These two metrics have more natural intuitive interpretations in R^2 than the metrics discussed in (1) and (2) and share some of the other properties of these metrics.

Simeon Reich, The Technion, writes regarding the paper *Open mappings and the fundamental theorem of algebra* by R. L. Thompson, pp. 39–40, January 1970, that the fact that the image of the plane by a nonconstant complex polynomial is open is an immediate consequence of the closedness of this image and of the connectedness of the plane. Let $p(z)$ be a nonconstant polynomial and let $w_n \to w$ where $w_n=p(z_n)$. Then $\{z_n\}$ must be bounded because $z_{n_k} \to \infty$

would imply $p(z_{n_k})\to\infty$. Thus $\{z_n\}$ has a convergent subsequence $z_{n_j}\to z$. Hence $p(z)=\lim_{j\to\infty} p(z_{n_j})=\lim_{j\to\infty} w_{n_j}=w$, as required.

In his article *Choreographic proof of a theorem on permutations*, May–June 1970, F. Cunningham, Jr., Bryn Mawr College, challenges the reader to present a direct proof of the theorem. Readers tempted to send us their response to this challenge should note that a very simple direct proof is available. It may be found, for example, in N. Bourbaki, *Éléments de Mathêmatique*, Livre II, Algèbre, Paris, p. 97, and in E. Schenkman, *Group Theory*, Van Nostrand, Princeton, p. 113.

Andrzej Makowski notes that the paper *A note on sums of squares of consecutive odd integers* by J. A. H. Hunter, May 1969, should have been listed as a reference in his paper on the same subject in the September 1970 issue. Also J. A. H. Hunter notes that in the case $n=241$, listed as unresolved in Makowski's paper, it can easily be shown that there is no solution.

R. G. Stanton, The University of Manitoba, notes that the substance of the article *Morley's triangle theorem* by R. J. Webster, p. 209, September 1970, is included in the article *The Morley triangle* by R. G. Stanton and H. C. Williams, p. 32 of the September 1965 issue of the Ontario Secondary School Mathematics Bulletin, and that this article also includes the expression for the distance from the vertices of the triangle as given in the article *Morley's triangle* by J. C. Burns which also appeared in the September 1970 issue of this MAGAZINE.

Larry Collister, William Rainer Harper College, writes with regard to the articles on the instant insanity puzzle that their computer successfully completed the search for solutions to the instant insanity puzzle in 17 minutes.

Kent MacDougall, West Texas State University, writes that Claude H. Raifaizen's *A simpler proof of Heron's formula*, p. 40, January 1971, is essentially the same as the solution of Problem 1.5–14(b), p. 40, *A Survey of Geometry*, by Howard Eves.

D. R. Byrkit, The University of West Florida, points out in connection with *Two problems on magic squares* by Elizabeth H. Agnew, January 1971, that by eliminating her condition 4, a total of 32 solutions may be obtained, and they are given in the Pi Mu Epsilon Journal for Fall 1968, pp. 383–384.

Dennis Anson, Western Kentucky University, and Aaron Strauss, University of Maryland, write: "We wish to express our strong disapproval of the policy that allowed the publication of the paper *Comments on a trajectory-indicating device* by J. L. Brenner, in MATHEMATICS MAGAZINE, volume 44 (1971), 92–94. In the first place a discussion involving 'hovering helicopter', 'attack from ground fire,' 'evasive action,' 'bullets were fired,' etc., does not help the reader understand any mathematical concepts. Secondly, the article contained absolutely no mathematics whatever. . . . everybody knows mathematics can be used for war (as Brenner's article illustrates). The question is, can mathematics be used for peace?"

M. G. Monzingo, Southern Methodist University, writes regarding his paper *On group elements of order two*, March 1971, that he has found a much shorter proof of his Theorem 1 and that he has also obtained the following additional results: (1) Given any odd integer n, there exists a group with exactly n elements of order two; (2) if G is abelian, then n is of the form 2^m-1; (3) given any odd integer of the form 2^m-1, there exists a group with exactly 2^m-1 elements of order two.

Leslie Shader, University of Wyoming, submitted affirmative answers to the two questions posed by Lyle Pursell in the article *Anti-isomorphisms vs. isomorphisms* in the March 1971 issue. The questions were: (1) Are there semigroups having no anti-automorphisms? (2) Is there a noncommutative semigroup A and a one-to-one mapping α from A onto A such that $\alpha(ab)$ is in $\{\alpha(a)\alpha(b), \alpha(b)\alpha(a)\}$ for all a, b in A but $\alpha(ab)=\alpha(a)\alpha(b)\neq\alpha(b)\alpha(a)$ and $\alpha(cd)=\alpha(d)\alpha(c)\neq\alpha(c)\alpha(d)$ for some a, b, c, d in A? He gives the 4 element semigroup A defined by the table

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d	d	c	c	d

as an example of a noncommutative semigroup satisfying (1). Then A with the mapping α defined by $\alpha(a)=a$, $\alpha(b)=b$, $\alpha(c)=d$, $\alpha(d)=c$ satisfies (2).

Answers to these two questions raised by Lyle Pursell were also submitted by David E. Manes, State University College, Oneonta, New York; Gilbert Steiner, Fairleigh Dickinson University; G. M. Leibowitz, University of Connecticut; Jiang Luh, North Carolina State University; and Robert Crawford, Western Kentucky University.

Erwin Just, Bronx Community College, writes that the argument given in *A combinatorial proof that* $\sum k^3=(\sum k)^2$, May 1971, by Robert G. Stein, appeared in the solution of Problem 3792 of the June 1938 issue of the AMERICAN MATHEMATICAL MONTHLY which was reprinted in the Otto Dunkel Memorial Problem Book. Professor M. G. Greening of the University of New South Wales informs us that the content of this paper is essentially equivalent to a problem set as a question in the Senior Division of the School Mathematics Competition run annually by the University of New South Wales in 1963.

THE 23 COLORED CUBES

N. T. GRIDGEMAN, Ottawa, Canada

Because of their ease of construction and perfect packability, cubes—and especially colored cubes—have always figured prominently in recreations and games [1]. Mathematically, coloring here means identity labeling, so that the "colors" may be numerals, letters, arbitrary symbols, patterns—or even colors.

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Because of their ease of construction and perfect packability, cubes—and especially colored cubes—have always figured prominently in recreations and games [1]. Mathematically, coloring here means identity labeling, so that the "colors" may be numerals, letters, arbitrary symbols, patterns—or even colors.

Given 6 colors we can create 2226 distinct cubes; this is an established result; but there is no agreed taxonomy within which this set can be broken down. Let us therefore introduce some *ad hoc* definitions and subdivisions that can lead to a rational classification:

A *family* is one of 6 primary subdivisions, according to the number of colors on the cube. For convenience we take 1 through 6 as the possible numbers, although in some contexts the number of *different* colors may have to be considered, in which circumstance the possible numbers will be 0, 2, 3, 4, 5, and 6.

A *genus* is conditioned by the frequency distribution of the colors on the cube. The distribution is easily symbolized. For instance, the genus of the family of twocolor cubes with 4 faces of one color (and necessarily 2 of the other) may be designated 420000—which can be coded down to 420. All genera (there are 11) will be similarly coded by 3 digits (the omitted triplet being inferable).

A *species* is determined by the type and number of planes of symmetry through the cube. There are 2 types: planes parallel to faces, of which there can be 0, 1, 2, or 3; and planes through face diagonals, of which there can be 0, 1, 2, or 6. The notation may be exemplified by "32," which means 3 planes of symmetry of the first type, and 2 of the second, on a particular cube. Species 00, uniquely, are enantiomorphic. In all there are 23 species.

A *variety* is a permutation of the given colors on a particular species.

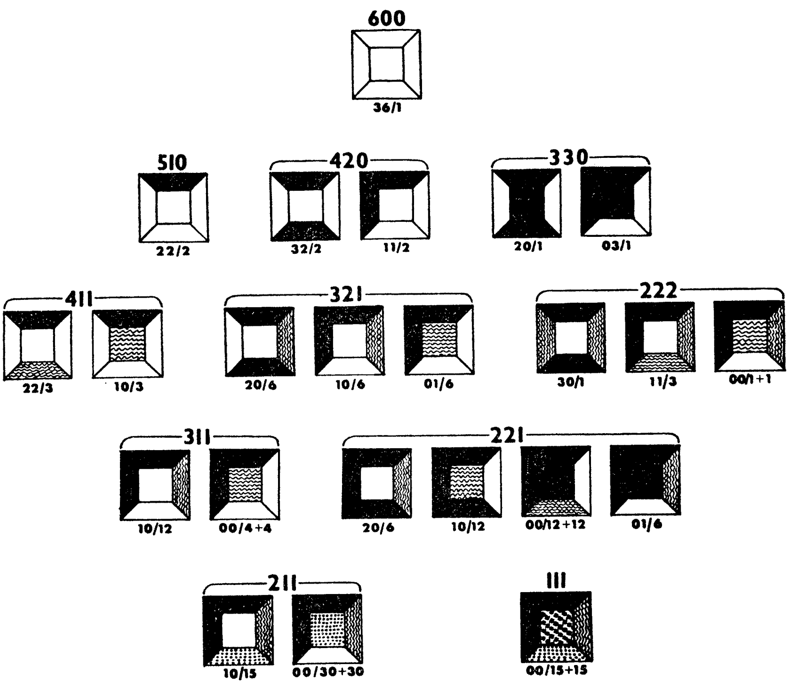

FIG. 1. Colored Schlegel diagrams of the 23 species of the colored cubes. The superior numbers denote the genera, and the inferior the species with (on the RHS of the solidus) the number of varieties. In all cases the unseen 6th side of the cube is colored white.

An *individual* is characterized by the actual colors on the cube, and this is bound up with the number of colors available—which can of course be more than 6. When the number of available colors *is* 6, the totality of individuals is 2226. For any species the number of individuals is

$$V\binom{C}{F}$$

where F is the family ($1 \leqq F \leqq 6$), and V the number of varieties, and C the colors available ($C \geqq F$).

Whereas family, genus, and species can be coded definitively, variety and individual cannot—unless the colors are assigned, or naturally possess, a rank order. However, the numbers of varieties and individuals that exist can usefully be indicated, and with this in mind we introduce a notation typified by

321-20/6

which specifies a threecolor cube, with the distribution 3, 2, and 1, having 2 planes of color symmetry of the first type, and none of the second; and finally we read that this species has 6 varieties. The whole scheme is illustrated in Figure 1, in which the 23 species are represented as colored Schlegel diagrams (which are also graphs). Observe that the number of varieties is given as a pair sum when the species is 00; this is to emphasize the enantiomorphism.

From this analysis it will be apparent that the intrinsic differentiation is that of *species*, and we can meaningfully speak, simply, of "the 23 colored cubes."

Many combinatoric problems arise in connection with colored cubes. Here we shall merely look at some aspects of the four- and six-color families, with special reference to questions of the arrangement of colors on the sides of aligned columns (single stacks) of cubes over a given set of colors.

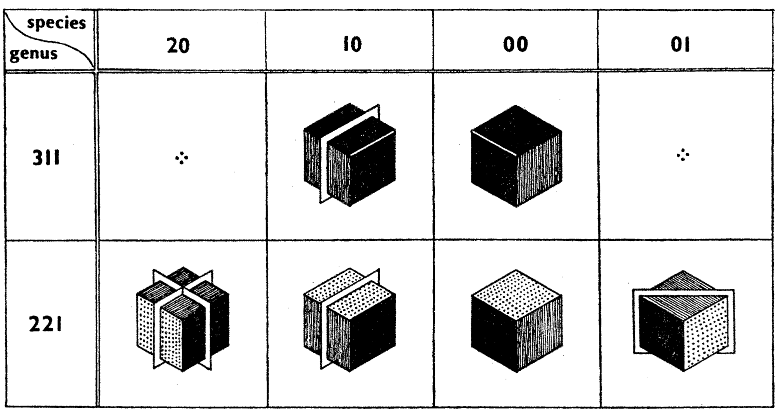

FIG. 2. Isometric drawings of the 6 fourcolor cubes with their planes of symmetry (which determine the species) highlighted. The colors of the unseen triad of faces (common to all 6 cubes) are unambiguously deducible from the visible information.

Stacked tetrads of fourcolor cubes. Evidently there are 68 varieties of the fourcolor cube. The 6 species are shown isometrically in Figure 2, with the genera—the dispositions of the planes of symmetry—highlighted. Symmetries are important in the makeup of a stack of cubes, any one of which has 24 orientations relative to a neighbor. On a stacked tetrad we shall speak of the 4 "show" faces and the 2 "contact" faces of any component cube. Now in the case of species 01 all 24 orientations are different in the sense that each exhibits a distinct set of show faces and contact faces. In species 10, one third of the orientations consists of 4 identical pairs as regards show faces, but not contact faces. Species 20 is similar to 10, except that the remaining two-thirds consists of identical pairs as regards show faces *and* contact faces, so that the effective number of orientations is 16 instead of 24. This is a rider to the fact that cubes with more than one plane of symmetry (of either kind) exhibit rotational symmetry. Finally, in species 00, all of whose orientations are entirely different, the two members of each pair of enantiomorphs are indistinguishable in stack context. This means that we can ignore half of the enantiomorphs (i.e., half of the varieties of species 00) in what follows. Thus the effective number of fourcolor cubes in all stacking problems is not 68 but 52 (which, interestingly, is the number of playing cards in a deck).

To check and familiarize yourself with the above statements, it is worthwhile constructing and handling the appropriate colored cubes.

The 52 fourcolor cubes will form

$$\binom{52}{4} = 270,725$$

different tetrads, each of which can be stacked a certain number of ways. But the stacking number is not the same for all tetrads, because it depends on how many representatives of species 20 are included, a reduction factor of 2/3 being here involved, as just pointed out. If m is the number of representatives of species 20 in the tetrad, and if we bear in mind that the "first" cube in an aligned tetrad can have only 3 orientations, the totality of stacks can be shown to be

$$\sum_{m=0}^{4} \binom{6}{m}\binom{46}{4-m} 3(4!)^3(2/3)^m = 7,\ 206,\ 955,\ 520$$

—regardless of the *order* of the 4 cubes.

Various features of such tetradic stacks can be considered in probabilistic terms or as combinatoric puzzle material. An attractive feature is to have all 4 show sides of the stack exhibit all 4 colors, and puzzle sets with this as solution are familiar.

How many solutions (stacks whose show sides each exhibit all 4 colors but whose contact sides are here irrelevant) exist among the more than 7 billion possible stacks? Let us think of the 16 show faces as a square matrix of order 4. Each of its rows must contain all 4 colors, and each column must contain at least 2 different colors (this follows from the genus characteristics). If we denote the colors as A, B, C, and D, the possible columns fall into 6 subsets, which are shown, together with their permutations and their distributions among species, in Table 1. Now it is apparent that any solution can be so matrixed that the

TABLE 1

Distribution of "show" faces per cube among the fourcolor cubes

Reference	I	II	III	IV	V	VI
Show columns	A B C D	A A B C	A B A C	A A B B	A B A B	A A A B
Permutations of colors	1	12	12	6	6	12
Genus and Species:						
311–10	1		1			1
311–00		3				
221–20			2		1	
221–10		1	2			
221–00	1	2				
221–01	2			1		
totals	4	6	5	1	1	1

TABLE 2

Types of solution to 4 fourcolor cube puzzles

First column	Restrictions on other columns	Number of fillings
I	none	216
II	I not allowed	156
III	I and II not allowed	51
IV	I thru III not allowed	7
V	I thru IV not allowed	1
VI	I thru V not allowed	3
	Totality of solution types	434

first row reads $ABCD$ and the first column is one or another of the 6 possibilities given at the head of Table 1. By studying the ways in which these matrix "frames" can be filled out (within the terms of the specification) we can arrive at the information required. The results are shown in Table 2. Let us be clear about the meaning of this totality of types: it is, that out of more than 7 billion fourcolor-cube puzzle sets, all solutions can be so arranged (by manipulation of cube order and designation of the "first" row) that they belong to one or more of these 434 types.

Two questions stand out among many that can be asked about fourcolor-cube puzzle sets. One is: How can solutions be found (by methods other than trial-and-error)? And the other is: What is the distribution of solutions-per-puzzle over the 270, 725 sets?

TABLE 3

Puzzle set I: (27,648 stacks; 1 solution)

Genus and species	331–10	331–10	221–20	221–10
Cubes	1 2 2 2 5 3	1 5 2 5 5 3	1 5 2 5 1 3	1 3 2 5 1 2
Opposite faces products	4–5–6	5–6–25	1–6–25	1–6–15
Solution search	4× 5= 20 4× 6= 24 4×25=100 5× 5= 25* 5× 6= 30 5×25=125 6× 5= 30 6× 6= 36** 6×25=150		900/ 1× 1=900 900/ 1× 6=150 900/ 1×15= 60 900/ 6× 1=150 900/ 6× 6= 25* 900/ 6×15= 10 900/25× 1= 36** 900/25× 6= 6 900/25×15= 12/5	
Solution:	1 2 3 2 5 2	5 5 2 5 1 3	3 1 5 1 2 5	2 3 1 5 3 1

The first question is the easier. Indeed several answers are known. The crudest is exhaustive checking, which is less onerous than might be expected (in view of the fact that the possibilities may exceed 40,000) because short-cut systematic search routes are not hard to devise. Smillie has programmed such a search for a digital electronic computer [2]. A more interesting device, pioneered by de Carteblanche [3], is based on graph-theoretic considerations. A puzzle set can be fully represented by a graph, and if, under certain conditions, this graph can be split into 2 subgraphs, a solution exists and its configuration is quickly determinable.

A third—and, surely, the best—approach is to focus attention on opposite pairs of faces, suitably symbolized, on a cube. In its original form, due to T. A. Brown [4], the idea is that the colors are represented by the noncomposite digits 1, 2, 3, and 5, so that all sets of faces, including of course, pairs, can be uniquely characterized by their *color products*. B. L. Schwartz [5] advocates an algebraic version of Brown's scheme in which the color key is in letters—the initial letters of the face colors. The two versions are mathematically equivalent, but Schwartz finds that the algebraic version is quicker in practice. The arithmetic version seems the better for exposition, so I use it here.

We concentrate on the color products of the 3 pairs of opposite faces on any one cube, seeking tetrads of pairs, within the puzzle set, whose product is 900 (that is, the square of $1 \times 2 \times 3 \times 5$), because if 2 such disjoint tetrads can be picked out, a solution has been reached. An example should illumine the subversion (of the arithmetic version) that I favor; see Table 3.

At the top of the table a puzzle set is depicted in an obvious two-dimensional schema. In the puzzle set as a whole there are of course $3^4 = 81$ four-face products, and these are most efficiently scanned in the form of two subsets of 9 products. In the center of the table the *products* associated with the first and second cubes are given on the left, and on the right are the *quotients* of the products of the third and fourth cubes divided into 900. We now merely have to pick out left-and-right like pairs, and it is clear that two such disjoint pairs exist (they are asterisked); therefore one and only one, solution exists. This solution is depicted at the bottom of the table—and here the 4-item column of "crosses" represent the (aligned) show faces, and the "arm" items represent the contact faces.

TABLE 4

Three other puzzle sets

Puzzle set II (27,648 stacks; 0 solutions)			
311–10	311–10	221–20	221–01
1 1 3 1 2 5	1 2 3 3 5 3	1 2 5 2 3 5	1 3 3 5 2 5
Puzzle set III (41,472 stacks; 1 solution)			
311–10	221–10	221–00	221–00
1 1 2 1 3 5	1 3 2 3 5 5	1 2 2 5 3 3	1 3 2 2 5 1
Puzzle set IV (41,472 stacks; 28 solutions)			
221–00	221–00	221–01	221–01
1 1 2 2 3 5	1 3 2 5 3 5	1 1 2 3 3 5	1 2 2 5 3 5

In Table 4 another 3 puzzle sets are shown, together with the number of solutions that exist, and the checking of these numbers can be an exercise. The middle set (III) is of uncommon note in that it is the basis of the most popular puzzle set: with 1 = red, 2 = white, 3 = green, and 5 = blue, it is the "Instant Insanity" puzzle now on wide sale, and many earlier models have been around under other names, such as the "Tantalizer." The last set (IV) in Table 4 has 28 solutions, which I think is the maximum (under the restriction that all 4 cubes are different and that enantiomorphic pairs are forbidden).

We now turn to the question of the distribution of solutions-per-puzzle, the extrema being zero and (probably) 28. As there are more than 250,000 puzzles an exact answer is not easy to obtain, but sampling-approximation is worthwhile. The Brown-Schwartz method of locating solutions is helpful here, as it lends itself to simple programming for electronic computation. This being done (IBM 360:FORTRAN IV), I drew a random sample of 100 puzzle sets (that is, tetrads taken at random, with replacement, from the population of 52 cubes), and ascertained the number of solutions to each. The results (which took just over a minute to obtain) are set out in Table 5. It appears then that about 40% of puzzle sets are in fact solutionless. Sets with one or two solutions make up about a third of the total. The average number of solutions per set in this sample is about 1.7.

TABLE 5

Distribution of numbers of solutions-per-puzzle in 100 random 4-cube puzzle sets

Number of solutions	Frequency
0	40
1	15
2	19
3	8
4	10
5	1
6	4
7	0
8	3
	100

Presumably, although by no means necessarily, the "Instant Insanity" puzzle set was arrived at empirically; the interesting thing is that hardly any attempt has been made to popularize any of the many other equally "puzzling" sets.

As a coda to the combinatorics of a puzzle set, we might look at the standard ("Instant Insanity") set more closely, in terms of special dice, with face colors numbered 1 through 4 (corresponding to red, white, green, and blue). Then the probabilities of each of the 13 possible results (as scores 4 through 16) in rolling the 4 dice together, are of interest. The relevant data and the probabilities are given in Table 6.

Sixcolor cubes. The existence of 30 varieties of this genus, all belonging to the one species, was published and discussed less than half a century ago by MacMahon [5]. A sizable literature has since accrued. There is an obvious link

TABLE 6

Rolling the 4 "standard" fourcolor cubes as dice

Score	Make-up, by pips	Ways	Probability
4	1111	6	0.004 630
5	1112	26	020 062
6	1113, 1122	67	051 697
7	1114, 1123, 1222	131	101 080
8	1124, 1133, 1223, 2222	196	151 235
9	1134, 1224, 1233, 2223	233	179 784
10	1144, 1234, 1333, 2224, 2233	226	174 383
11	1244, 1334, 2234, 2333	181	139 660
12	1344, 2244, 2334, 3333	122	094 136
13	1444, 2344, 3334	67	051 697
14	2444, 3344	29	022 377
15	3444	10	007 716
16	4444	2	0.001 543
		1296	1

with dice, whose colors are 6 sets of pips. Therefore 30 different dice exist, but with the traditional restriction that opposite faces should have a common sum (7 pips, necessarily), only 2 varieties can be used. They are most readily distinguished by reference to the corner shared by pip sets 1, 2, and 3, whose order may be clockwise or counterclockwise. The latter is the modern standard arrangement.

The set of 30 consists of 15 pairs of enantiomorphs, and the makeup of this set can be visualized as follows: Color A can have any one of the remaining 5 colors on the opposite face; then the remaining 4 colors are disposable in a ring, regardless of direction, in 3 different ways. This accounts for the 15. Distinguish between the 2 possible directions; then the 30 are accounted for.

Because of the absence of duplicate faces, the stacking combinatorics of such cubes is more straightforward and less interesting than that of the fourcolor cubes. Let us briefly consider stacks of 6. If enantiomorphic pairs are disallowed in the hexad, the number of possible puzzle sets is

$$\binom{15}{6} 2^6 = 320,\ 320$$

each of which can be stacked in

$$3(4!)^5 = 23,\ 887,\ 872$$

different ways. Therefore the number of puzzle sets is the product of these two numbers, namely,

$$7,\ 651,\ 763,\ 159,\ 040.$$

Given any puzzle set, and numbering the colors 1, 2, 3, 5, 7, and 11, or denoting them by initial letters, we can use the Brown-Schwartz method to find solutions —a solution being a stack that exhibits all 6 colors on each of the 4 sides.

Finally, let us take a quick look at the question of how many solutions exist for a puzzle set of 6 *identical* cubes (and in this context a pair of enantiomorphs is an identical pair). In other words, in how many ways can we stack 6 standard dice so that each of the stack sides displays all 6 sets of pips?

Consider the representation of the 24 show faces of the stack as a 4×6 matrix. Then the problem reduces to finding the number of ways that the digits 1, 3, 4, 5, and 6 can replace the letters V, W, X, Y, and Z in the following matrix to form a Latin Rectangle:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & V & W & X & Y & Z \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & (7-V) & (7-W) & (7-X) & (7-Y) & (7-Z) \end{matrix}$$

The answer is easily shown to be 20. This matrix is cast in the form of a particular die (or, at least, of a particular enantiomorphic pair of dice among the 15 possibilities), but the result applies to all dice.

Note added in proof. The "Instant Insanity" puzzle is also discussed by Chiao Yeh in *The American Statistician*, 24 (1970) No. 3, p. 13, by some letter-to-the-editor writers in *The American Statistician*, 24 (1970) No. 5, pp. 37–38, and by A. P. Grecos and R. W. Gibberd, this MAGAZINE, 44 (1971) 119–124.

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SOME EXTENSIONS OF NIM

BENJAMIN L. SCHWARTZ, Mathematica

Dedicated to Miriam Levine

1. Introduction. The primary purpose of this note is to give a solution to a mathematical game believed previously unsolved. The game is one of the family described as match pile games. The familiar game of nim is a member of this family, and indeed the game to be solved can be considered a variant of nim. We shall call it bounded nim.

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A secondary purpose is to describe a further natural extension of nim, bounded extended nim, which is an unsolved problem that has received some attention lately. We shall challenge the reader to attack it, and show certain difficulties that will lie in his path.

2. Match pile games. In Fred Schuh's encyclopedic *Master Book of Mathematical Recreations*, (Dover, 1968) Chapter VI provides an intensive discussion of match pile games. The following summary is based on Schuh's description.

One or more piles of matches are provided. Alfred and Benjamin take turns removing one or more matches according to definite rules which constitute the rules of the particular game. In this paper, we follow the convention that whoever takes the last match is the winner. (The alternate form in which the player taking the last match loses is entirely similar.)

There are certain positions (defined by the numbers of matches in the separate piles) in which the player who has just finished his turn can win, if he continues correctly. These are called winning positions. All others are called losing positions. The set of winning positions is characterized by the following three properties:

1. The final position (no matches in any pile) is winning.
2. From a winning position, any move permitted by the rules of the game creates a losing position.
3. From a losing position, there is always (at least) one permitted move that creates a winning position. (The description of this move is called the winning strategy.)

3. Examples. To clarify the ideas, we shall describe concretely some match pile games with known solutions, i.e., the winning strategies have been completely identified.

A. Battle of Numbers.

Description. There is one pile of matches. At each turn, a player may take any number (but at least one) up to an agreed upon maximum, k.

Solution. The winning positions are found by reducing the number of matches in the pile modulo $(k+1)$. If the result is 0, the position is winning; otherwise losing. The winning strategy is therefore to take enough matches so that the pile contains an exact multiple of $(k+1)$.

This result is well known, and the simple proof is given in many places, including Schuh's book. We omit this derivation, since it is so easily available elsewhere.

B. Nim.

Description. Another familiar game is the following. The number of piles is arbitrary. Say initially there are m piles. At a player's turn, he may choose at will any one pile. He then may take from that pile any number of matches, but at least one.

Since an entire pile can be taken, the number of piles will decrease as play proceeds. At the end there will be *no* piles.

Solution. In this game also, the winning positions are well known, although more difficult to characterize. Express the contents of each pile as a binary number. Align the several binary numbers so obtained as though for addition. However, do not add them. (That would merely result in the binary representation of the total number of matches in all piles.) Instead, count the number of "1" bits in each column. If every column has an even number of "1's," the position is winning; otherwise losing.

This result too is proved in many places, including Schuh, and hence not proved here.

4. Bounded nim. The match pile game now to be described and solved is a natural combination of the two foregoing familiar games. In bounded nim, there is initially any number of piles, say m. At a player's turn, he may select any (nonempty) pile at will. He may then take from it a number of matches; at least one, but not more than a previously agreed upon maximum, k.

This game is considered by Schuh with special values of m and k. For values of k up to 5, he gives the winning strategies. They are rather complicated to describe. For larger values of m and k, Schuh characterizes this game as an especially difficult problem.

But in fact it is a simple thing to give the solution for general m and k, simply combining the rules of the preceding two games.

Solution. Reduce the number of matches in each pile modulo $(k+1)$. Express the resulting set of residue numbers (each less than $k+1$) in binary form; and align these binary numbers as though for addition. In each column, add up the number of "1" bits. If all these sums are even, then the position is winning; otherwise losing.

Example. Let there be four piles of 75, 54, 95, and 110 matches respectively. Let $k=10$. The procedure is then to reduce the numbers of matches in the piles modulo 11, obtaining the residues 9, 10, 7, and 0 respectively. In binary representation, these are

$$\begin{array}{llll} 1 & 0 & 0 & 1, \\ 1 & 0 & 1 & 0, \\ 0 & 1 & 1 & 1, \text{ and} \\ 0 & 0 & 0 & 0. \end{array}$$

The column sums are 2, 1, 2, and 2. Since one of these is odd, the given position is losing. It can be converted to a winning position by eliminating the one "1" digit in the second column from the left. This will make the column total 0, which is even. The binary representation of the third pile must become 0 0 1 1, which corresponds to 3. By taking four matches from the pile of 95, we would leave 91 matches in that pile. Since $91 \equiv 3$ (modulo 11), this is the winning play.

5. Proof of the solution. Since this winning strategy has not been previously published, a proof is called for.

First, the final position (all zeroes) certainly is in the set, since the binary representation of 0 consists of all zeroes. There will be no "1" bits; and all column sums will be zero, hence even.

Secondly, suppose a winning position has been attained by Alfred at the end of his play. Then when Benjamin plays, he will necessarily change one pile; and its binary representation will be changed in at least one column, a "1" to a "0" or vice versa. In bounded nim, the binary representation can either increase or decrease. This is unlike ordinary nim, where only a decrease is possible. Of course, Benjamin's move necessarily decreases the total count of matches in some pile. But the residue modulo $(k+1)$ can increase. For example, if $k=10$ and a pile has 110 matches, its residue modulo 11 is zero. Any removal of 10 matches or less will create an increase in the residue. But the important fact is that the representation must *change*. The only way a pile might be reduced without changing the residue modulo $(k+1)$ is if it were reduced by a multiple of $k+1$. But this would be forbidden by the rules, since k is the upper limit on the number of matches that may be taken.

Finally, if Alfred leaves a losing position, then Benjamin can change one of the binary representations of a pile residue to a *smaller* number to attain a winning position. Benjamin does this by treating the residues as if they described a position in ordinary nim. If the residues (all less than $(k+1)$) were actually pile sizes in ordinary nim, a play would be available in that game of ordinary nim creating a winning position, i.e., all column sums even. This would require removing matches from one pile, and the number to be removed would not be more than k, since the entire pile would not have more than k matches. When Benjamin determines the pile and the correct number of matches to take from that pile that he would play in ordinary nim, he then uses the same move in bounded nim. This will create a winning position in which all the column sums in the binary representation are even. This completes the proof.

In view of the simple and straightforward way the winning strategy can be obtained by adapting from the solutions of the other two earlier games, the author finds it surprising that previous work has only dealt laboriously with special cases.

Yet it does not always follow that because a game is a simple extension of another with known solution, the solution to the extension follows easily from the other. Our next examples illustrate this point.

6. Extended nim. Another game considered by Schuh and others is a variant of nim described as follows. The number of piles is initially arbitrary, say, m. At a player's turn to play, he selects at will any number of *piles* up to an agreed upon maximum of n, but at least one. Then he takes matches from each pile selected, at least one from each, with no upper limit except the pile size. We shall call this game extended nim.

Solution. This game was first described and solved by E. H. Moore in 1910. The solution with derivation is given by Schuh and in various other places. The winning strategy is as follows.

The numbers of matches in the piles are expressed in binary. The resulting numbers are aligned, and the separate column sums determined. If each column sum is a multiple of $(n+1)$, then the position is winning; otherwise losing.

The modification of the winning strategy when nim is changed to extended nim is so slight as to be almost trivial. Instead of column sums being multiples of 2, they must be multiples of $(n+1)$. Clearly nim and extended nim are closely related games. One might conjecture that extended nim could also be played in a bounded form, and the previous winning strategy for bounded nim could be readily adapted to solve bounded extended nim. That conjecture might be reasonable, but it is completely wrong! The new game does not yield to any such elementary approach. Indeed, the solution is believed unknown.

7. Bounded extended nim. For definiteness, we define the game. There is initially an arbitrary number of piles, say m. At a player's turn, he selects at least one, and up to as many as n piles, where n is a previously agreed upon bound. From each pile selected, he takes at least one and not more than k matches, where k is another agreed upon bound. The winner (as in all match pile games considered in this paper) is the player taking the last match.

As stated in the previous section, for the general game, the winning strategy is not known. In fact, the special cases that have been solved are few. The simplest possible bounds are $n=2$, $k=1$. In this game, a player takes either one match; or two matches, one from each of two different piles. Even for this very simple case, solutions have been found only for values of m up to 5.

Partial solution. For the case of $n=2$, $k=1$, $m=5$, let the piles be ordered, left to right, in sequence of increasing (or nondecreasing) size. Then the winning positions depend only upon the parity of the piles. We write "e" for even, and "u" for odd (uneven). The winning positions are:

$$e \quad e \quad e \quad e \quad e,$$
$$e \quad e \quad u \quad u \quad u,$$
$$u \quad u \quad u \quad u \quad e, \quad \text{and}$$
$$u \quad u \quad e \quad e \quad u.$$

These solutions also apply to four piles (or less) by considering the smallest pile(s) as zero, hence even.

These results are again in Schuh, although without proofs. They were apparently independently rediscovered recently. The solutions, with detailed derivations, are in a lively article in a recent issue of the *Journal of Recreational Mathematics* by F. De Carte Blanche [**5**].

8. Conclusion: a challenge. The reader is challenged to find more results about bounded extended nim. It appears that this will not be easy. There are striking dissimilarities between the solutions for the special cases where answers are known, and all the other games discussed in this article. In the former, the order in which the piles are treated is important. They must be in increasing order from left to right. The winning position $e\ e\ u\ u\ u$ is not merely one with two even and three odd piles; it is also necessary that the two even piles must be smaller than the three odd piles. Thus 2, 4, 5, 5, 9 is winning; but 2, 5, 5, 6, 9 is not. In fact, faced with the latter position, Alfred can play to leave Benjamin the former.

Moreover, in the solved cases of bounded extended nim, the binary representation plays little role—only the parity of the piles is important. (Of course, in the binary representation, the units position differentiates between even and odd. But in ordinary nim, it is essential to consider *all* positions of the binary representation, not just the units position.)

The author's search for a general solution (or at least, a more general solution) to bounded extended nim has been fruitless. However, he has been able to show one negative result. Whatever the general solution is, it cannot be characterized solely by the order of the piles and their parity.

Consider the first unsolved case: $n=2$, $k=1$, $m=6$. In particular, for this game, consider the position left by Alfred 1, 1, 1, 1, 1, 1. It is easy to verify that this is a winning position. If Benjamin plays from two piles (and necessarily exhausts them), then Alfred will play from one; and vice versa. The parity representation of this winning position is $u\ u\ u\ u\ u\ u$. But positions with this representation can also be losing. For example, consider the position 1, 1, 1, 1, 1, 3. If Alfred leaves this position, then Benjamin can move to leave the 5-pile position (0), 1, 1, 1, 1, 2, which is one of the winning positions ($u\ u\ u\ u\ e$) with five piles.

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COVERING CLASSES OF RESIDUES IN $Z(\sqrt{-2})$

J. H. JORDAN and D. G. SCHNEIDER, Washington State University

1. Introduction. A set of ordered pairs of integers $\{(a_i, m_i)\}$ is said to cover the integers if each integer x satisfies the congruence $x \equiv a_i \pmod{m_i}$ for some i. A finite set of ordered pairs that covers the integers and has the additional properties that all $m_i > 1$ and $m_i \neq m_j$ for $i \neq j$ is called a covering class of residues. P. Erdös [1] introduced the concept and exhibited the examples $\{(0, 2), (0, 3), (1, 4), (7, 8), (11, 12), (19, 24)\}$ and $\{(0, 2), (0, 3), (1, 4), (5, 6), (7, 12)\}$. Erdös [2] investigated the additional condition that $m_i > n$ and exhibited an example for $n=2$. Swift [6] and Jordan [3] added examples for $n=3$. Selfridge [5] found an example for $n=7$.

Erdös [2] posed the question "For each positive integer n is there a covering class of residues with the property that each modulus exceeds n?" He offered a $50 reward for a positive or a negative answer to the question.

The Gaussian integers are those complex numbers $\beta = a+bi$ where a and b are real integers.

Recently Jordan [3] defined a covering class of residues in the Gaussian integers to be a finite set of ordered pairs of Gaussian integers, $\{(a_j, \gamma_j)\}_{j=1}^{m}$, with $|\gamma_j| > 1$ and for $j \neq k$, $\gamma_j \neq \gamma_k \epsilon$, where $\epsilon = \pm 1$ or $\pm i$, and having the property the binary representation, the units position differentiates between even and odd. But in ordinary nim, it is essential to consider *all* positions of the binary representation, not just the units position.)

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Let $\theta = \sqrt{-2}$. The integers of $Q(\theta)$ are those complex numbers $a+b\theta$ where a and b are real integers. Call this set of integers $Z(\theta)$.

Divisibility is defined on $Z(\theta)$ as follows:

DEFINITION 1. *For α, γ in $Z(\theta)$, $\gamma \neq 0$, $\gamma | \alpha$ means there is a δ in $Z(\theta)$ such that $\gamma\delta = \alpha$.*

The congruence relation is meaningful in $Z(\theta)$ and is defined by

DEFINITION 2. *$\alpha \equiv \beta \pmod{\gamma}$ means $\gamma | \alpha - \beta$.*

The analogy of covering classes of residues can now be extended to $Z(\theta)$ by defining:

DEFINITION 3. *A covering class of residues in $Z(\theta)$ is a finite set of ordered pairs of elements of $Z(\theta)$, $\{(\alpha_j, \gamma_j)\}_{j=1}^{m}$, with $|\gamma_j| > 1$ and $\gamma_j \neq \pm\gamma_k$, such that every β in $Z(\theta)$ satisfies $\beta \equiv \alpha_j \pmod{\gamma_j}$ for some j.*

The purpose of this paper is to establish the existence of infinitely many covering classes of residues in $Z(\theta)$ and pose the Erdös-like restriction and give examples for $|\gamma_j|^2 > n$ for $n = 2$, 3, 4 and 5.

2. An elementary example. To better illustrate the concept and to show a style of proof we establish:

EXAMPLE 1. *The set $\{(0, \theta), (1, 2), (0, 1+\theta), (3+\theta, 2-\theta), (5+\theta, 2+2\theta)\}$ is a covering class of residues in $Z(\theta)$.*

Proof. Consider the element $a+b\theta$ with a even. Then

$$a + b\theta = \theta(-\theta\, a/2 + b),$$

hence $a+b\theta \equiv 0 \pmod{\theta}$.

If a is odd and b is even, then $a+b\theta-1 = 2((a-1)/2 + b\theta/2)$, hence $a+b\theta \equiv 1 \pmod{2}$.

If $a \equiv b \equiv s \pmod{3}$ where $0 \leq s < 3$ then

$$\begin{aligned} a + b\theta - s(1+\theta) &= (a-s) + (b-s)\theta \\ &= (1+\theta)(1-\theta)((a-s)/3 + (b-s)\theta/3) \end{aligned}$$

or equivalently

$$a + b\theta = (1+\theta)(s + (1-\theta)((a-s)/3 + (b-s)\theta/3)).$$

Therefore $a+b\theta \equiv 0 \pmod{1+\theta}$.

If a and b are both odd then for any c and d that are odd one has

$$a + b\theta - (c + d\theta) = 2((a - c)/2 + (b - d)\theta/2)$$

or $a+b\theta \equiv c+d\theta \pmod 2$ and consequently $a+b\theta \equiv c+d\theta \pmod \theta$.

If $a \equiv b-1 \equiv s \pmod 3$, $0 \leqq s < 3$, then

$$\begin{aligned} a + b\theta - (3 + \theta) - s(1 + \theta) &= a - 3 - s + (b - 1 - s)\theta \\ &= 3((a - 3 - s)/3 + (b - 1 - s)\theta/3), \end{aligned}$$

hence $a+b\theta \equiv 3+\theta+s(1+\theta) \pmod 3$ and consequently $a+b\theta \equiv 3+\theta \pmod{1+\theta}$. Since θ and $1+\theta$ are relatively prime in $Z(\theta)$ and 3 and 1 are odd $a+b\theta \equiv 3+\theta \pmod{2-\theta}$.

Now finally if $a \equiv b+1 \equiv s \pmod 3$, $0 < s < 3$, then

$$\begin{aligned} a + b\theta - (5 + \theta) + (2 + 2\theta) - s(1 + \theta) &= a - 3 - s + (b + 1 - s)\theta \\ &= 3((a - 3 - s)/3 + (b + 1 - s)\theta/3) \end{aligned}$$

or equivalently

$$a + b\theta = 5 + \theta + (1 + \theta)\{-2 + s + (1 - \theta)((a - 3 - s)/3 + (b + 1 - s)\theta/3)\}.$$

Hence $a+b\theta \equiv 5+\theta \pmod{1+\theta}$ and since 2 and $1+\theta$ are relatively prime and 5 and 1 are odd it follows that $a+b\theta \equiv 5+\theta \pmod{2+2\theta}$.

All possibilities for a and b have been considered so the set mentioned does cover $Z(\theta)$.

A symmetric elementary example can be obtained by replacing the given set by the set with conjugate elements as entries in the ordered pairs. The above proof goes through by making minor adjustments.

3. An infinitude of essentially different covers. A covering class of residues in $Z(\theta)$, $A = \{(\alpha_j, \gamma_j)\}$, is said to be essentially different from the covering class of residues in $Z(\theta)$, $B = \{(\beta_j, \eta_j)\}$, if (i) there is a γ_j that is prime and $\gamma_j \nmid \eta_k$ for any k, (ii) there does not exist a covering class $A^1 = \{(\alpha_k, \gamma_k)\}_{k \neq j}$.

To establish an infinitude of essentially different covering classes of residues in $Z(\theta)$ we proceed as follows:

LEMMA 1.

$$\theta^{n-1} \equiv 2^{[n/2]} + 2^{[(n-1)/2]}\theta \pmod{\theta^n}, \tag{1}$$

$n = 1, 2, \cdots$, *where* [ ] *is the greatest integer function.*

Proof. If n is even, say $n = 2t$, then the right-hand side of (1) is

$$\begin{aligned} 2^t + 2^t\theta &= (-\theta^2)^t + (-\theta^2)^{t-1}\theta \\ &= (-1)^t\theta^n + (-1)^t\theta^{n-1} \equiv (-1)^t\theta^{n-1} \pmod{\theta^n}. \end{aligned}$$

Because $-1 \equiv 1 \pmod \theta$, we have $-\theta^{n-1} \equiv \theta^{n-1} \pmod{\theta^n}$, so whether t is odd or even, we have the desired result. If n is odd, the verification is similar.

LEMMA 2. *If* $\beta \not\equiv 0 \pmod{\theta^m}$, *then* $\beta \equiv 2^{[n/2]} + 2^{[(n-1)/2]}\theta \pmod{\theta^n}$ *for some* n, $n = 1, 2, \cdots, m$, *where* [ ] *is the greatest integer function.*

Proof. We proceed by induction. When $m=1$, the lemma states that if $\beta \not\equiv 0 \pmod{\theta}$, then $\beta \equiv 1 \pmod{\theta}$, which is true. Suppose that the lemma is true for $m=k$. Suppose that $\beta \not\equiv 0 \pmod{\theta^{k+1}}$. There are two cases: either $\beta \not\equiv 0 \pmod{\theta^k}$ or $\beta \equiv 0 \pmod{\theta^k}$. In the first case, the induction assumption applies, and the conclusion of the lemma is true. In the second case, we have

$$\beta = (c + d\theta)\theta^k = c\theta^k + d\theta^{k+1}, \tag{2}$$

and since $\beta \not\equiv 0 \pmod{\theta^{k+1}}$, we know that $c \not\equiv 0 \pmod{\theta}$. Thus $c \equiv 1 \pmod{\theta}$, and from (2), $\beta \equiv \theta^k \pmod{\theta^{k+1}}$. Applying Lemma 1, the proof is complete.

Two special cases of results established by Potratz [4] are:

LEMMA 3. *If $\gamma = a + b\theta$, $ab \neq 0$, is a prime in $Z(\theta)$ then $\{0, 1, 2, \cdots, a^2+2b^2-1\}$ is a complete residue system modulo γ.*

LEMMA 4. *If p is a prime of $Z(\theta)$ and a real prime then*

$$\{x + y\theta\colon 0 \leqq x < p,\ 0 \leqq y < p\}$$

is a complete residue system modulo p.

A straightforward generalization of a result in the real case is,

LEMMA 5. *If α and δ are relatively prime in $Z(\theta)$ then $\{\alpha_\eta + \beta\}$, where η ranges over a complete residue system modulo δ, is a complete residue system modulo δ.*

The proof follows the pattern of the real case.

We are now ready to exhibit the parametrizations that establish the infinitude of essentially different covering classes of residues.

THEOREM 1. *If $\gamma \neq \pm\theta$ is a nonreal prime of $Z(\theta)$ with $p = |\gamma|^2$ and if $A = \{(2^{[n/2]} - 1 + (2^{[(n-1)/2]} - 1)\theta, \theta^n)\}_{n=1}^{p-1}$ and $B = \{(n2^{(p-1)/2} - 1 - \theta, \theta^n\gamma)\}_{n=0}^{p-1}$ then $A \cup B$ is a covering class of residues in $Z(\theta)$.*

THEOREM 2. *If p is a real prime and a prime of $Z(\theta)$, if A^* is as in Theorem 1 only with the n ranging from 1 to p^2-1, and if*

$$C = \{(x + y\theta)2^{(p^2-1)/2} - 1 - \theta, \theta^{Z_{x,y}}p)\}_{z=0}^{p-1}{}_{y=0}^{p-1},$$

where $Z_{x,y}$ is arbitrary in the interval $[1, \cdots, p^2-1]$ and $Z_{x,y} = Z_{t,u}$ if and only if $x=t$ and $y=u$. Then $A^* \cup C$ is a covering class of residues in $Z(\theta)$.

Proof of Theorem 1. If n satisfies none of the congruences of A, then the contrapositive of Lemma 2 applied to $\eta+1+\theta$ says that $\eta \equiv 0 \pmod{\theta^{p-1}}$ or $\eta = \alpha 2^{(p-1)/2} - 1 - \theta$. By Lemma 3 $\alpha \equiv n \pmod{\gamma}$ for some $0 \leqq n \leqq p-1$ so $\eta \equiv n2^{(p-1)/2} - 1 - \theta \pmod{\gamma}$ and from above $\eta \equiv n2^{(p-1)/2} - 1 = \theta \pmod{\theta^n}$. Since θ^n and γ are relatively prime, we have by the Chinese Remainder theorem that $\eta \equiv n2^{(p-1)/2} - 1 - \theta \pmod{\theta^n}$ which is one of the congruences in set B.

The infinitude is of course established by the infinitude of primes as parameters.

The proof of Theorem 2 is similar and is omitted.

4. The Erdös question. We pose the question analogous to the one Erdös asked: "For every positive integer n does there exist a covering class of residues, $A=\{(\alpha_j,\gamma_j)\}$, $Z(\theta)$ such that $|\gamma_j|^2>n$ for all j?"

We offer a \$25 reward for the positive answer to the question and a \$75 reward for the negative answer.

We varied the amount of the rewards from the Erdös offer because it seems to us easier to attack the positive answer in $Z(\theta)$ than in Z and therefore harder to attack the negative part. What may be true is that all questions of this type will be answered positively or negatively together.

We have some examples we list without verification.

THEOREM 3. *The set* $\{(0,1+\theta),(0,1-\theta),(0,2),(1,3),(2,2-\theta),(1+\theta,2+\theta),(3+\theta,3\theta),(2+\theta,2+\theta)\}$ *is a covering class of residues with all* $|\gamma_j|^2>2$.

THEOREM 4. *The set* $\{(1+\theta,2),(0,2-\theta),(3,2+\theta),(2\theta,4+\theta),(1,4-\theta),(4,1-2\theta),(4,1+2\theta),(2,3),(4\theta,3-3\theta),(-4\theta,3+3\theta),(1+\theta,6+3\theta),(2+2\theta,6-3\theta),(4-3\theta,9\theta),(7,9),(3,2+2\theta),(-1,2-2\theta),(2+2\theta,2+4\theta),(4+3\theta,18)\}$ *is a covering class of residues with all* $|\gamma_j|^2>3$.

THEOREM 5. *If* $(1+\theta,2)$ *is eliminated from the set of Theorem* 4 *and* $(5+\theta,2-4\theta)$ *and* $(1+\theta,6)$ *are added, then the resulting set will be a covering class of residues in* $Z(\theta)$ *with all* $|\gamma_j|^2>5$.

Research for this paper was supported in part by NSF Grant No. GP-6227, WSU Grants-in-aid for research, project number 728, and NSF Grant No. G-22765/0009 (undergraduate research).

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ON THE NINE POINT CONIC ASSOCIATED WITH A COMPLETE QUADRANGLE

FRANCIS REGAN and R. L. WILKE, St. Louis University

1. Introduction. It is a well-known proposition in affine geometry that the midpoints of the six sides of a complete quadrangle and the three diagonal points lie on a conic. When the quadrangle is orthocentric, the nine-point conic is a circle. An orthocentric complete quadrangle is one in which two pairs of opposite sides are perpendicular.

Here we explore by methods of analytic geometry the nature of this nine point conic associated with the complete quadrangle $ABCP$, where A, B, and asked: "For every positive integer n does there exist a covering class of residues, $A = \{(\alpha_j, \gamma_j)\}$, $Z(\theta)$ such that $|\gamma_j|^2 > n$ for all j?"

We offer a \$25 reward for the positive answer to the question and a \$75 reward for the negative answer.

We varied the amount of the rewards from the Erdös offer because it seems to us easier to attack the positive answer in $Z(\theta)$ than in Z and therefore harder to attack the negative part. What may be true is that all questions of this type will be answered positively or negatively together.

We have some examples we list without verification.

THEOREM 3. *The set* $\{(0, 1+\theta), (0, 1-\theta), (0, 2), (1, 3), (2, 2-\theta), (1+\theta, 2+\theta), (3+\theta, 3\theta), (2+\theta, 2+\theta)\}$ *is a covering class of residues with all* $|\gamma_j|^2 > 2$.

THEOREM 4. *The set* $\{(1+\theta, 2), (0, 2-\theta), (3, 2+\theta), (2\theta, 4+\theta), (1, 4-\theta), (4, 1-2\theta), (4, 1+2\theta), (2, 3), (4\theta, 3-3\theta), (-4\theta, 3+3\theta), (1+\theta, 6+3\theta), (2+2\theta, 6-3\theta), (4-3\theta, 9\theta), (7, 9), (3, 2+2\theta), (-1, 2-2\theta), (2+2\theta, 2+4\theta), (4+3\theta, 18)\}$ *is a covering class of residues with all* $|\gamma_j|^2 > 3$.

THEOREM 5. *If* $(1+\theta, 2)$ *is eliminated from the set of Theorem* 4 *and* $(5+\theta, 2-4\theta)$ *and* $(1+\theta, 6)$ *are added, then the resulting set will be a covering class of residues in* $Z(\theta)$ *with all* $|\gamma_j|^2 > 5$.

Research for this paper was supported in part by NSF Grant No. GP-6227, WSU Grants-in-aid for research, project number 728, and NSF Grant No. G-22765/0009 (undergraduate research).

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ON THE NINE POINT CONIC ASSOCIATED WITH A COMPLETE QUADRANGLE

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Here we explore by methods of analytic geometry the nature of this nine point conic associated with the complete quadrangle $ABCP$, where A, B, and

C are noncollinear fixed distinct points and P is any point on the plane such that no three of the four points are collinear. The sets of points P are determined for which the conic will be an ellipse, a circle, and a hyperbola.

2. The general conic associated with its complete quadrangle. Let us be given any complete quadrangle $ABCP$ on the Cartesian plane. By a transformation of the axes and appropriate lettering of the triangle ABC, we may take $A(0, 0)$, $B(x_0, 0)$, and $C(x_1, y_1)$, where $x_1 < x_0$, and x_1, y_1 and x_0 are positive. Let P have coordinates (x, y), and it now becomes our problem to discuss what type conic is connected with the quadrangle $ABCP$ when $P(x, y)$ takes different positions on the plane. (See Figure 1.)

First, it is easily shown that the orthocenter of the triangle ABC is $H(x_1, x_1(x_0-x_1)/y_1)$, from which follows that the nine-point conic of the complete quadrangle $ABCH$ is a circle (this will be shown later). The equation of the line through the points $P(x, y)$ and $C(x_1, y_1)$ is

$$y - y_1 = m(x - x_1).$$

Thus the coordinates of P are $(x, mx - mx_1 + y_1)$. Construct the complete quadrangle $ABCP$. For the nine-point conic, the points to be considered are the diagonal points $D = BC \cap AP$, with x coordinate

$$\frac{-xx_0y_1}{mxx_1 - mx_1^2 - mxx_0 + mx_0x_1 - y_1x_0 - y_1x + y_1x_1}$$

and y coordinate

$$\frac{-x_0y_1(mx - mx_1 + y_1)}{mxx_1 - mx_1^2 - mxx_0 + mx_0x_1 - y_1x_0 - y_1x + y_1x_1},$$

$E = AC \cap BP$ with x coordinate

$$\frac{mxx_0x_1 - mx_1^2x_0 + x_0y_1x_1}{-y_1x + y_1x_0 + mxx_1 - mx_1^2 + y_1x_1}$$

and y coordinate

$$\frac{mxx_0y_1 - mx_1x_0y_1 + x_0y_1^2}{-y_1x + y_1x_0 + mxx_1 - mx_1^2 + y_1x_1},$$

and $F = CP \cap AB$, with x coordinate

$$\frac{-mx_1 + y_1}{-m}$$

and y coordinate 0.

Also to be considered are the midpoints of the six sides AB, AC, BC, BP, AP, and CP, which have coordinates $G(x_0/2, 0)$, $J(x_1/2, y_1/2)$, $K(\{x_1+x_0\}/2, y_1/2)$, $M(\{x_0+x\}/2, \{mx-mx_1+y_1\}/2)$, $N(x/2, \{mx-mx_1+y_1\}/2)$, and $R(\{x_1+x\}/2, \{mx-mx_1+2y_1\}/2)$, respectively.

In order to show that these nine points lie on a conic, select any four of these points, say J, K, M, and N.

Let $L_1=0$, $L_2=0$, $L_3=0$, and $L_4=0$ be the equations of the lines JK, MN, KN, KM, respectively, with variables x', y'.

Form equations $L_1L_2+kL_3L_4=0$. It is easily shown that the equation for the locus becomes

$$
\begin{aligned}
&y'^2 + y'\frac{(mx_1 - mx - 2y_1)}{2} - \frac{(y_1mx_1 - mxy_1 - y_1^2)}{4}\\
(1)\qquad &+ k\Big\{y'^2 - 2mx'y' + y'(mx_0 + 2mx_1 - 2y_1) + m^2x'^2\\
&- mx'\frac{(mx_0 + 2mx_1 - 2y_1)}{2} + \frac{(mx_1 - y_1)(mx_0 + mx_1 - y_1)}{4}\Big\} = 0.
\end{aligned}
$$

Substituting $G(x_0/2, 0)$, one of the remaining five points, in equation (1) yields

$$
(2)\qquad k = \frac{y_1(mx_1 - mx - y_1)}{-m^2x_1x_0 + mx_0y_1 + m^2x_1^2 - 2mx_1y_1 + y_1^2}.
$$

Substituting this value of k in equation (1) gives an equation of a conic through the five points J, K, M, N, and G. Designate this as equation (2). All four of the remaining points D, E, F, and R satisfy equation (2).

To determine the type of conic in (2) we apply the usual criterion. One considers

$$
(3)\qquad \frac{4m^2y_1^2(mx_1-mx-y_1)^2}{\{-m^2x_1x_0+mx_0y_1+m^2x_1^2-2mx_1y_1+y_1^2\}^2} - \frac{4m^2y_1(mx_1-mx-y_1)(-m^2x_1x_0+mx_0y_1+m^2x_1^2-2mx_1y_1+y_1mx_1-mxy_1)}{\{-m^2x_1x_0+mx_0y_1+m^2x_1^2-2mx_1y_1+y_1\}^2}.
$$

Simplifying and dropping the positive denominator, the numerator with the deletion of the factor 4, becomes

$$
(4)\qquad (mx_1 - mx - y_1)\left(m - \frac{y_1}{x_1}\right)\left(m + \frac{y_1}{x_0 - x_1}\right).
$$

Consider $\infty > m > y_1/x_1$ and the ordinate of point $P(x, mx-mx_1+y_1)$ as positive. Then the point $P(x, mx-mx_1+y_1)$ is chosen in regions I and II of Figure 1. The sign of (4) is negative and hence the nine-point conic is an ellipse.

Let $\infty > m > y_1/x_1$ and $(mx-mx_1+y_1)<0$. This places $P(x, mx-mx_1+y_1)$ in region III, Figure 1. The sign of (4) is positive so the nine-point conic is a hyperbola.

Let $y_1/(x_1-x_0) > m > -\infty$ and $(mx-mx_1+y_1)>0$. Then $P(x, mx-mx_1+y_1)$ lies in regions IV and V of Figure 1. Then (4) is negative, thus the nine-point conic is an ellipse.

Let $y_1/(x_1-x_0)>m>-\infty$ and $(mx-mx_1+y_1)<0$. $P(x, mx-mx_1+y_1)$ lies in region VI. The factors in (4) are $(+)$ $(-)$ $(-)$ making the product positive, hence the nine-point conic is a hyperbola.

Let $0<m<y_1/x_1$ and $(mx-mx_1+y_1)>0$. This requires that $P(x, mx-mx_1+y_1)$ lies in regions VII and VIII. The sign of the product in (4) is positive, hence the nine-point conic is a hyperbola.

Let $0<m<y_1/x_1$ and $(mx-mx_1+y_1)<0$, then $P(x, mx-mx_1+y_1)$ lies in region IX. The sign of (4) is negative so the nine-point conic is an ellipse.

Let $m=0$, the sign of (4) is positive, hence the nine-point conic is a hyperbola. It should be noted that when $m=0$, P becomes (x, y_1) and lies on the line $y=y_1$, where x cannot equal x_1, for if it did, there would be no complete quadrangle.

Let $0>m>y_1/(x_1-x_0)$ and $(mx-mx_1+y_1)>0$. Here $P(x, mx-mx_1+y_1)$ lies in regions X and XI. Expression (4) is positive, with the nine-point conic being a hyperbola.

Let $0>m>y_1/(x_1-x_0)$ and $(mx-mx_1+y_1)<0$. Then $P(x, mx-mx_1+y_1)$ lies in region XII of Figure 1, (4) is negative, and the nine-point conic is an ellipse.

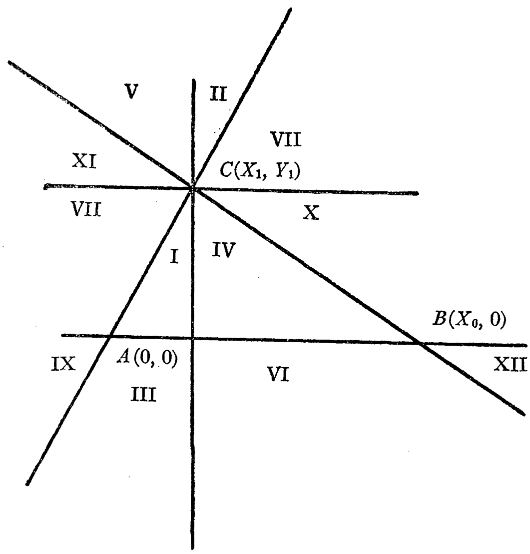


FIG. 1.

All points $P(x, mx-mx_1+y_1)$ are taken care of except those lying on line $x=x_1$. Let us now take care of this case.

Construct a complete quadrangle using vertices $A(0, 0)$, $B(x_0, 0)$, $C(x_1, y_1)$, and $P(x, y)$ where $x=x_1+k(x_1-x_1)$, $y=y_1+k(x_1x_0-x_1^2-y_1^2)/y_1$, and $x_1<x_0$, and x_1, y_1, x_0 are positive. Thus $P(x, y)=P(x_1, y_1+k(x_1x_0-x_1^2-y_1^2)/y_1)$ which is a point lying on the altitude through $C(x_1, y_1)$. (Figure 1.)

The nine points to be considered are the diagonal points $D=BC\cap AP$, with x abscissa

$$\frac{-x_0x_1y_1^2}{2kx_1^2x_0-kx_1^3-ky_1^2x_1-x_0y_1^2-kx_1x_0^2+ky_1^2x_0}$$

and ordinate

$$\frac{(y_1^2 + k(x_1x_0 - x_1^2 - y_1^2))(-x_0y_1)}{2kx_1^2x_0 - kx_1^3 - ky_1^2x_1 - x_0y_1^2 - kx_1x_0^2 + ky_1^2x_0}.$$

$E = CA \cap BP$, with x abscissa

$$-\left(\frac{-x_1x_0y_1^2 - kx_1^2x_0^2 + kx_1^3x_0 + ky_1^2x_1x_0}{y_1^2x_0 + kx_1^2x_0 - kx_1^3 - ky_1^2x_1}\right)$$

and y ordinate

$$-\left(\frac{-x_0y_1^3 - kx_1x_0^2y_1 + kx_1^2x_0y_1 + ky_1^3x_0}{y_1^2x_0 + kx_1^2x_0 - kx_1^3 - ky_1^2x_1}\right)$$

and $F = AB \cap CP$ with x coordinate x_1 and y coordinate 0.

Also to be considered are the midpoints of the six sides AB, AC, BC, BP, AP, and CP, which have coordinates $(x_0/2,\ 0)$, $(x_1/2,\ y_1/2)$, $(\{x_1+x_0\}/2,\ y_1/2)$, $(\{x_0+x_1\}/2,\ \{y_1^2+k(x_1x_0-x_1^2-y_1^2)\}/2y_1)$, $(x_1/2,\ \{y_1^2+k(x_1x_0-x_1^2-y_1^2)\}/2y_1)$, and $(x_1,\ \{2y_1^2+k(x_1x_0-x_1^2-y_1^2)\}/2y_1)$, respectively.

To determine the conic on which the nine points lie, select four of the nine points, for example, $E(x_1/2,\ y_1/2)$, $F(\{x_1+x_0\}/2,\ y_1/2)$, $G(\{x_1+x_0\}/2,\ \{y_1^2+k(x_1x_0-x_1^2-y_1^2)\}/2y_1)$, $H(x_1/2,\ \{y_1^2+k(x_1x_0-x_1^2-y_1^2)\}/2y_1)$, and form equation $L_1L_2+kL_3L_4=0$, where $L_1=0$, $L_2=0$, $L_3=0$, and $L_4=0$ are the equations of the lines through HG, EF, HE, and GF, respectively.

It is easily verified that $L_1L_2+kL_3L_4=0$ becomes

$$\begin{aligned} &y^2 + y\{(-2y_1^2 - k(x_1x_0 - x_1^2 - y_1^2)\}/2y_1) \\ &+ \{y_1^2 + k(x_1x_0 - x_1^2 - y_1^2)\}/4 \\ &+ k'\{x^2 + [(-2x_1 - x_0)/2]x + x_1(x_1 + x_0)/4\} = 0. \end{aligned} \tag{5}$$

Substituting any of the remaining points in equation (5), for example $(x_0/2,\ 0)$ gives

$$k' = \frac{-y_1^2 - k(x_1x_0 - x_1^2 - y_1^2)}{x_1^2 - x_0x_1}. \tag{6}$$

Substituting the value k' of (6) in equation (5) gives

$$\begin{aligned} &y^2 + \{y(-2y_1^2 - k(x_1x_0 - x_1^2 - y_1^2))\}/2y_1 \\ &+ \{y_1^2 + k(x_1x_0 - x_1^2 - y_1^2)\}/4 + [\{-y_1^2 - k(x_1x_0 - x_1^2 - y_1^2)\}/(x_1^2 - x_0x_1)] \\ &[x^2 + \{(-2x_1 - x_0)/2\}x + \{x_1(x_1 + x_0)\}/4] = 0. \end{aligned} \tag{7}$$

It is an easy matter to verify that the remaining four points satisfy equation (7). To determine the type of conic in (7), using the usual criterion, one obtains

$$\{4(-y_1^2 - k(x_1x_0 - x_1^2 - y_1^2))\}/(-x_1^2 + x_1x_0). \tag{8}$$

Simplifying and dropping the positive denominator of the expression (8) gives the pertinent part as

$$-y_1^2 - k(x_1x_0 - x_1^2 - y_1^2). \tag{9}$$

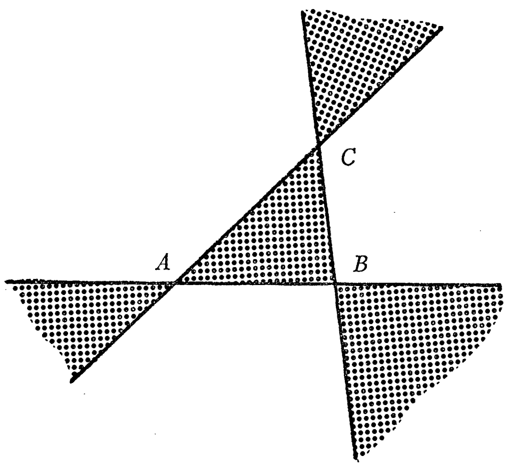
If $k=0$, $P(x_1, y_1+\{k(x_1x_0-x_1^2-y_1^2)\}/y_1)=P(x_1, y_1)$, and hence we have no complete quadrangle.

If $k=1$, then

$$P(x_1, \{y_1^2 + k(x_1x_0 - x_1^2 - y_1^2)\}/y_1) = P(x_1, \{x_1(x_0 - x_1)\}/y_1) = H$$

(the orthocenter of triangle ABC) and equation (7) is a circle, for the coefficients of x and y are 1 and there is no xy term.

If the ordinate of $P(x_1, \{y_1^2+k(x_1x_0-x_1^2-y_1^2)\}/y_1)$ is greater than zero, then (9) is negative and the conic is an ellipse. If the ordinate is zero, there is no complete quadrangle. If the ordinate is less than zero, then (9) is positive and the nine point conic is a hyperbola.



FIG, 2.

In conclusion, one may say that given any three distinct noncollinear points A, B, C, a point P may be chosen with no three points collinear to form a complete quadrangle. If $P(x, y)$ is selected in the shaded portion (Figure 2), then the nine-point conic will be an ellipse. When $P(x, y)$ is the orthocenter H, the nine-point conic is a circle. If $P(x, y)$ is chosen in the clear section, then the nine-point conic will be a hyperbola.

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AN INTEGRAL INEQUALITY

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The purpose of this brief note is to record the following useful inequality whose proof is a simple application of the Cauchy-Schwarz inequality (see this MAGAZINE, 42 (1969) p. 162). To our knowledge, the result is not in the literature in the generality stated here, and possibly will be of interest to several readers. Our terminology follows that of [1].

THEOREM. *Let* $(X, \mathfrak{M}, \mu)$ *be a measure space, and* $f: X \to (0, \infty]$ *a measurable function. Then*

$$\left(\int_X f d\mu\right)\left(\int_X 1/f d\mu\right) \geqq \mu(X)^2.$$

Typical applications of the theorem are:

COROLLARY 1. *For any* $t \geqq 0$, $(\int_0^t e^{x^2} dx)(\int_0^t e^{-x^2} dx) \geqq t^2$.

COROLLARY 2. *If* $a_1, a_2, \cdots, a_n$ *are positive real numbers, then*

$$\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n a_i^{-1}\right) \geqq n^2.$$

COROLLARY 3. *If* Y *is a positive random variable and* $E(Y) < \infty$, *then* $E(1/Y) \geqq 1/E(Y)$.

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A PROOF THAT NOT BOTH πe AND $\pi + e$ ARE ALGEBRAIC

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It is an unsolved problem of algebra whether the numbers πe and $\pi + e$ are transcendental or algebraic. In this note we prove that at least one of the two is transcendental.

Let us assume that both πe and $\pi + e$ are algebraic. Then using the fact that the algebraic numbers form a field so that we may do multiplication, division by nonzero numbers, addition, and subtraction on them, we observe that $(\pi+e)^2 - 4\pi e$ is algebraic. Now $(\pi+e)^2 - 4\pi e = \pi^2 - 2\pi e + e^2 = (\pi - e)^2$, so $(\pi - e)^2$ is algebraic. Since the roots of a polynomial equation with algebraic coefficients are algebraic, the roots of $x^2 - (\pi - e)^2 = 0$ are algebraic. Then $(\pi + e) + (\pi - e) = 2\pi$ is algebraic, which we know is not true. Therefore, at least one of πe and $\pi + e$ must be transcendental.

By a tedious series of similar computations, we can show that no more than one member of the list of $\pi+e$, $\pi - e$, πe, $\pi^2 + e^2$, and $\pi^2 - e^2$ can be algebraic. The method is to consider each pair and show that not both can be algebraic.

ON A MOMENT PROBLEM

M. Z. NASHED, University of Wisconsin, Madison

Suppose that a rigid rod of unit mass and unit length is allowed to oscillate in a plane as a pendulum about one end as the point of suspension. If c is a given real number, is it possible to prescribe the mass distribution of the rod (call it $f(x)$) so that (i) f is a continuous function on $[0, 1]$, (ii) the center of mass is c distant from the point of suspension, and (iii) the period of small oscillation about the equilibrium position is $2\pi\sqrt{c/g}$, where g is the acceleration due to gravity?

It is easy to show that the equation of the oscillation of the pendulum is given by

$$\ddot{\theta} + g\alpha^2 \sin\theta = 0, \quad \text{where} \quad \alpha^2 = \int_0^1 xf(x)dx \Big/ \int_0^1 x^2 f(x)dx,$$

from which it follows that the above problem is equivalent to the following moment problem:

Does there exist a continuous nonnegative function f such that

$$(1) \quad \int_0^1 f(x)dx = 1, \quad (2) \quad \int_0^1 xf(x)dx = c, \quad \text{and} \quad (3) \quad \int_0^1 x^2 f(x)dx = c^2?$$

We present four solutions of this problem.

1. Suppose there exists a continuous nonnegative function satisfying (1)–(3). Multiplying (1), (2) and (3) by c^2, $-2c$ and 1 respectively and adding we get $\int_0^1 (x-c)^2 f(x)dx = 0$, which implies that f is the zero function. This contradicts (1); hence there exists no function with the properties stated in the problem.

2. A closely related solution is based on the Parallel Axes theorem for second moments of mass distribution: $I_y = I_G + Mc^2$, where I_G is the moment of inertia with respect to an axis parallel to the y-axis and passing through the center of mass. Substituting we get $I_G = 0$. But no mass distribution can have a moment of inertia which is zero. It should be noted that solutions 1 and 2 are not completely independent of each other, since if one wishes to derive the Parallel Axes theorem, one would be expanding $\int_0^1 (x-c)^2 f(x)dx$. Another equivalent approach is to use a probabilistic interpretation of (1)–(3).

We now show how the necessary and sufficient conditions for *equality* in Cauchy's and Jensen's inequalities can be invoked in the setting of the moment problem.

3. If u and v are continuous real functions on $[0,1]$, then

$$(4) \qquad \left(\int_0^1 u(x)v(x)dx\right)^2 \leqq \int_0^1 u^2(x)dx \int_0^1 v^2(x)dx,$$

with the equality holding if and only if u and v are linearly dependent on $[0, 1]$. With reference to our problem, we let $u(x) = \sqrt{f(x)}$, $v(x) = x\sqrt{f(x)}$. Then

$$\left(\int_0^1 xf(x)dx\right)^2 = c^2 = \int_0^1 f(x)dx \int_0^1 x^2f(x)dx,$$

which shows that the equality holds in (4). But the functions $\sqrt{f(x)}$ and $x\sqrt{f(x)}$ are linearly independent for a nonzero function f. Thus we arrive again at a contradiction.

4. Let g be a strictly convex function on $(-\infty, \infty)$, i.e., for all x and y, $x \neq y$, and any positive numbers α, β with $\alpha+\beta=1$, $g(\alpha x+\beta y)<\alpha g(x)+\beta g(y)$. Then for each x_0, there exists a number $\lambda(x_0)$ such that

$$g(x) \geqq g(x_0) + \lambda(x_0)(x - x_0), \tag{5}$$

with the equality holding only if $x=x_0$. We remark that (5) characterizes convex functions and that $\lambda(x_0)$ is the one-sided directional derivative at x_0, i.e., $\lambda(x_0)h=\lim_{t\to 0^+} (1/t)\{f(x_0+th)-f(x_0)\}$.

Now let u be any continuous function and f be a nonnegative function such that $\int_0^1 f(x)dx=1$. It follows from (5) that

$$g(u(t)) \geqq g\left(\int_0^1 u(t)f(t)dt\right) + \lambda\left(u(t) - \int_0^1 u(t)f(t)dt\right).$$

Multiplying both sides of the last inequality by $f(t)$ and integrating over $[0, 1]$, we get

$$g\left(\int_0^1 u(t)f(t)dt\right) \leqq \int_0^1 g(u(t))f(t)dt. \tag{6}$$

This is the form of Jensen's inequality that we need. Note that equality in (6) holds if and only if u is constant. Now letting $g(x)=x^2$ and $u(x)=x$ in (6), we obtain

$$\left(\int_0^1 tf(t)dt\right)^2 \leqq \int_0^1 t^2f(t)dt. \tag{7}$$

If there exists a function f which satisfies the properties given in the moment problem, then equality holds in (7). This would imply that u is constant, which is not the case.

SHADOWS OF FOUR-DIMENSIONAL POLYTOPES

BRUCE L. CHILTON, State University College at Fredonia, New York

The symbol $\{p, q\}$ denotes the regular polyhedron bounded by p-gons, q at each vertex. When p and q are integers, the polyhedron is convex. There are five such polyhedra, namely $\{3, 3\}$, $\{4, 3\}$, $\{3, 4\}$, $\{5, 3\}$, and $\{3, 5\}$, respectively the tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

$$\left(\int_0^1 xf(x)dx\right)^2 = c^2 = \int_0^1 f(x)dx \int_0^1 x^2 f(x)dx,$$

which shows that the equality holds in (4). But the functions $\sqrt{f(x)}$ and $x\sqrt{f(x)}$ are linearly independent for a nonzero function f. Thus we arrive again at a contradiction.

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The symbol $\{p, q, r\}$ denotes the regular polytope in four dimensions bounded by polyhedra $\{p, q\}$, r sharing each edge. Again, $\{p, q, r\}$ is convex when p, q, and r are integers, and there are six such polytopes, namely:

$\{3, 3, 3\}$, bounded by 5 tetrahedra, with 10 faces, 10 edges, and 5 vertices;
$\{3, 3, 4\}$, bounded by 16 tetrahedra, with 32 faces, 24 edges, and 8 vertices;
$\{4, 3, 3\}$, bounded by 8 cubes, with 24 faces, 32 edges, and 16 vertices;
$\{3, 4, 3\}$, bounded by 24 octahedra, with 96 faces, 96 edges, and 24 vertices;
$\{3, 3, 5\}$, bounded by 600 tetrahedra, with 1200 faces, 720 edges, and 120 vertices; and
$\{5, 3, 3\}$, bounded by 120 dodecahedra, with 720 faces, 1200 edges, and 600 vertices.

Fascinating as these four-dimensional regular figures are, one is prevented from knowing them as intimately as one can come to know the regular polyhedra by the extreme difficulty of "visualizing" them. One way to facilitate such visualization is suggested by a method which we use every day to visualize three-dimensional objects, namely, projection of the objects into planes. In the same manner, it is possible to project a four-dimensional object into a hyperplane. We shall consider only the convex hulls of such projections. If one imagines such a convex hull as the boundary of a solid object, one can easily see the appropriateness of the name "shadow" for such a figure, by analogy with shadows of three-dimensional objects.

The shadows of four-dimensional objects which we shall consider arise by projection from a "light source at infinity", i.e., orthogonal projection normal to the hyperplane containing the shadow. Furthermore, we shall be interested only in shadows which possess a substantial degree of symmetry. Except for shadows of $\{3, 3, 3\}$, every shadow of a regular polytope $\{p, q, r\}$ possesses central symmetry; we shall require more symmetry than this. It is possible to obtain highly-symmetrical shadows by correct positioning of the polytope to be projected. There are four basic positions for each polytope which (again, except for $\{3, 3, 3\}$) yield four such shadows. Let m be the line joining the "light source at infinity" to the center of the polytope. We define four "special" shadows:

VERTEX-FIRST SHADOW: The shadow obtained when m contains a vertex of the polytope.

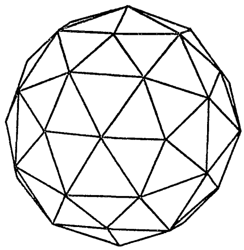
EDGE-FIRST SHADOW: m contains the midpoint of an edge of the polytope.

FACE-FIRST SHADOW: m contains the center of a face.

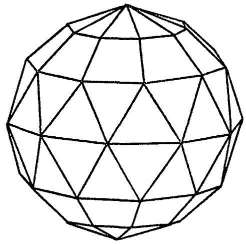
CELL-FIRST SHADOW: m contains the center of a cell (i.e., bounding polyhedron).

Many of the shadows presented here can be found using Table V in [1]. Four of them, the edge-first and face-first shadows of $\{3, 3, 5\}$ and $\{5, 3, 3\}$, are new. The following information describes each shadow, and gives complete sets of vertices in the cases where the shadows are not well-known polyhedra. Each ordered triple represents several vertices, obtainable from that triple by changing signs in all possible ways: for example, the triple (a, b, c) would yield vertices (a, b, c), $(a, b, -c)$, $(-a, b, -c)$, etc. In some cases, permutations of the given triples are to be taken also.

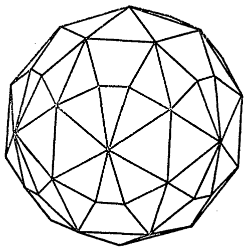
Where symmetry is described in terms of some well-known polyhedron, it is



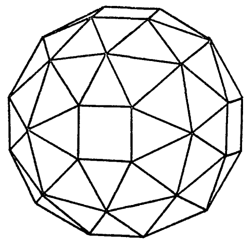
{3,3,5} V.F.



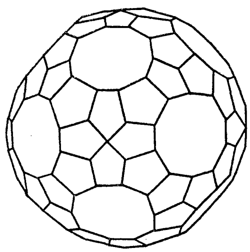
{3,3,5} E.F.



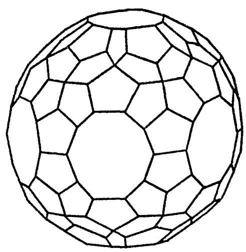
{3,3,5} F.F.



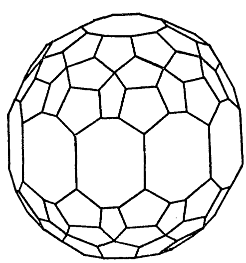
{3,3,5} C.F.



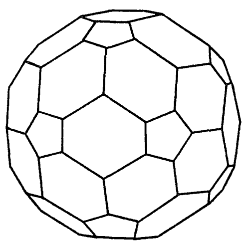
{5,3,3} V.F.



{5,3,3} E.F



{5,3,3} F.F.



{5,3,3} C.F.

to be assumed that this polyhedron has the greatest possible degree of symmetry for its type. For example, if a shadow is said to have the symmetry of a hexagonal prism, this means a right prism with regular hexagonal bases.

1a. Cell-first and vertex-first (which are identical) of a $\{3, 3, 3\}$ of **edge 1:** Regular tetrahedron of edge 1.

1b. Edge-first and face-first (also identical) of a $\{3, 3, 3\}$ of edge 1: Triangular dipyramid (triangles isosceles and congruent); equatorial edge 1, distance between apices 1. Has symmetry of the triangular prism.

2a. Cell-first of a $\{3, 3, 4\}$ of edge 2: Cube of edge $\sqrt{2}$.

(*Note.* Whenever a shadow of a polytope $\{p, q, r\}$ has a face with more than p sides, this face is the image of an entire cell of the polytope. Thus, in the present case, 6 of the 16 tetrahedra project into squares.)

2b. Vertex-first of a $\{3, 3, 4\}$ of edge 1: Regular octahedron of edge 1.

2c. Edge-first of a $\{3, 3, 4\}$ of edge $2\sqrt{2}$: Square dipyramid: $(\sqrt{2}, 0, 0)$, $(0, 0, 2)$, $(0, 2, 0)$.

2d. Face-first of a $\{3, 3, 4\}$ of edge $2\sqrt{2}$: Hexagonal dipyramid: $(2, 0, 0)$, $(0, 0, 4\gamma)$, $(0, \sqrt{2}, 2\gamma)$, where $\gamma = 6^{-1/2}$.

3a. Cell-first of a $\{4, 3, 3\}$ of edge 1: Cube of edge 1.

3b. Vertex-first of a $\{4, 3, 3\}$ of edge 2: Rhombic dodecahedron of edge $\sqrt{3}$.

3c. Edge-first of a $\{4, 3, 3\}$ of edge 3: Hexagonal right prism (regular bases); base edge $\sqrt{6}$, height 3.

3d. Face-first of a $\{4, 3, 3\}$ of edge 1: Square right prism; base edge 1, height $\sqrt{2}$.

4a. Cell-first of a $\{3, 4, 3\}$ of edge 1: Cuboctahedron of edge 1.

4b. Vertex-first of a $\{3, 4, 3\}$ of edge 2: Rhombic dodecahedron of edge $\sqrt{3}$.

4c. Edge-first of a $\{3, 4, 3\}$ of edge 2: $(2, 0, 0)$, $(1, 0, 4\gamma)$, $(1, \sqrt{2}, 2\gamma)$, where $\gamma = 6^{-1/2}$. Symmetry is that of the hexagonal prism.

4d. Face-first of a $\{3, 4, 3\}$ of edge $2\sqrt{2}$: $(2, \sqrt{2}, 2\gamma)$, $(2, 0, 4\gamma)$, $(0, \sqrt{2}, 6\gamma)$, $(0, 2\sqrt{2}, 0)$. Symmetry is that of the hexagonal prism.

5a. Cell-first of a $\{3, 3, 5\}$ of edge $2\tau^{-1}\sqrt{2}$, where $\tau = \frac{1}{2}(\sqrt{5}+1)$.

(*Note.* Here all permutations are to be taken of the ordered triples.) $(\tau^2, \tau^{-1}, \tau^{-1})$, $(2, 2, 0)$, (τ, τ, τ). Symmetry is that of the cube.

5b. Vertex-first of a $\{3, 3, 5\}$ of edge $2\tau^{-1}$. Here even permutations are to be taken: $(2, 0, 0)$, $(\tau, 1, 0)$, $(\tau, \tau^{-1}, 1)$. Symmetry is that of the icosahedron. In fact, this is an icosidodecahedron with a pentagonal pyramid attached to each pentagonal face.

5c. Edge-first of a $\{3, 3, 5\}$ of edge $2\tau^{-1}$, with $\beta = 5^{-1/4}\tau^{-5/2}$: $(2, 0, 0)$, $(\tau, 0, 2\beta\tau^2)$, $(\tau, \tau^{-1}, \beta\tau^3)$, $(\tau, 1, \beta\tau)$, $(1, 0, 2\beta\tau^3)$, $(1, 1, \beta\tau^4)$, $(1, \tau, \beta\tau^2)$, $(0, \tau^{-1}, \beta\tau^3\sqrt{5})$, $(0, \tau, \beta\tau^2\sqrt{5})$, $(0, 2, 0)$. Symmetry is that of the decagonal prism.

5d. Face-first of a $\{3, 3, 5\}$ of edge $2\tau^{-1}$, with $\alpha = 3^{-1/2}\tau^{-3}$: $(2, 0, 0)$, $(\tau, 0, 2\alpha\tau^3)$, $(\tau, 1, \alpha\tau^3)$, $(1, \tau^{-1}, \alpha\tau^5)$, $(1, 1, \alpha\tau^3\sqrt{5})$, $(1, \tau, \alpha\tau)$, $(\tau^{-1}, 0, 2\alpha\tau^4)$, $(\tau^{-1}, \tau, \alpha\tau^4)$, $(0, 1, 3\alpha\tau^3)$, $(0, 2, 0)$. Symmetry is that of the hexagonal prism.

6a. Cell-first of a $\{5, 3, 3\}$ of edge $2\tau^{-2}$. Even permutations are to be taken. $(\tau^2, \tau^{-2}, 1)$, $(\tau^2, 1, 0)$, $(\sqrt{5}, \tau^{-1}, \tau)$, $(2, 2, 0)$, (τ, τ, τ). Symmetry is that of the icosahedron. This figure can be described as a rhombic triacontrahedron that has been truncated at its vertices of pentagonal symmetry.

6b. Vertex-first of a $\{5, 3, 3\}$ of edge $2\tau^{-2}\sqrt{2}$, with $\sigma = \frac{1}{2}(3\sqrt{5}+1)$ and $\sigma' = \frac{1}{2}(3\sqrt{5}-1)$. All permutations are to be taken. $(4, 0, 0)$, $(\sigma, \tau^{-1}, \tau^{-1})$,

$(\tau\sqrt{5}, \tau, \tau^{-2})$, $(2\tau, 2, 2\tau^{-1})$, $(\tau^2, \tau^2, \tau^{-1}\sqrt{5})$, $(\sqrt{5}, \sqrt{5}, \sqrt{5})$. Symmetry is that of the cube.

6c. Edge-first of a $\{5, 3, 3\}$ of edge $2\tau^{-2}$, with $\alpha=3^{-1/2}\tau^{-3}$: $(\tau^2, \tau^{-2}, \alpha\tau^4)$, $(\tau^2, \tau^{-1}, \alpha\tau^2\sqrt{5})$, $(\tau^2, 1, \alpha)$, $(\sqrt{5}, \tau^{-1}, \alpha\tau^5)$, $(\sqrt{5}, 1, \alpha\tau^3\sqrt{5})$, $(\sqrt{5}, \tau, \alpha\tau)$, $(2, 0, 2\alpha\tau^4)$, $(2, 1, 3\alpha\tau^3)$, $(2, \tau, \alpha\tau^4)$, $(2, 2, 0)$, $(\tau, 0, 4\alpha\tau^3)$, $(\tau, \tau^{-1}, \alpha\tau^2(2\tau^2+1))$, $(\tau, \tau, \alpha(\tau^5+1))$, $(\tau, 2, 2\alpha\tau^3)$, $(\tau, \sqrt{5}, \alpha\tau^3)$, $(1, 0, \alpha(\tau^6+1))$, $(1, 1, \alpha\tau^6)$, $(1, \tau, \alpha\tau^4\sqrt{5})$, $(1, \sqrt{5}, \alpha\tau^3\sqrt{5})$, $(1, \tau^2, \alpha\tau^2)$, $(\tau^{-1}, \tau^{-1}, \alpha\tau(\tau^5+1))$, $(\tau^{-1}, 2, 2\alpha\tau^4)$, $(\tau^{-1}, \tau^2, \alpha\tau^2\sqrt{5})$, $(0, \tau^{-2}, 3\alpha\tau^4)$, $(0, \sqrt{5}, 3\alpha\tau^3)$, $(0, \tau^2, 3\alpha\tau^2)$. Symmetry is that of the hexagonal prism.

6d. Face-first of a $\{5, 3, 3\}$ of edge $2\tau^{-2}$, with $\beta=5^{-1/4}\tau^{-5/2}$: $(\tau^2, 0, 2\beta\tau^2)$ $(\tau^2, \tau^{-1}, \beta\tau^3)$, $(\tau^2, 1, \beta\tau)$, $(\sqrt{5}, 0, 2\beta\tau^3)$, $(\sqrt{5}, 1, \beta\tau^4)$, $(\sqrt{5}, \tau, \beta\tau^2)$, $(2, \tau^{-1}, \beta\tau^3\sqrt{5})$, $(2, \tau, \beta\tau^2\sqrt{5})$, $(2, 2, 0)$, $(\tau, \tau^{-2}, \beta\tau^5)$, $(\tau, 1, \beta\tau^2(3\tau-1))$, $(\tau, \tau, 3\beta\tau^2)$, $(\tau, 2, 2\beta\tau^2)$, $(\tau, \sqrt{5}, \beta\tau)$, $(1, \tau^{-1}, 3\beta\tau^3)$, $(1, 1, \beta\tau^2(\tau+3))$, $(1, 2, 2\beta\tau^3)$, $(1, \sqrt{5}, \beta\tau^4)$, $(1, \tau^2, \beta)$, $(\tau^{-1}, 0, 2\beta\tau^4)$, $(\tau^{-1}, \tau, \beta\tau^5)$, $(\tau^{-1}, \tau^2, \beta\tau^3)$. Symmetry is that of the decagonal prism.

Reference

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ON $\sum_{n=1}^{\infty}(1/n^{2k})$

KENNETH S. WILLIAMS, Carleton University, Ottawa

In this note we give a simple proof of the well-known result ([**1**], [**3**])

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\frac{2^{2k-1}\pi^{2k}B_k}{(2k)!},\qquad k=1, 2, 3, \cdots,$$

where B_k is the kth Bernoulli number, defined by

$$\sum_{k=1}^{\infty}B_k\frac{x^{2k}}{2k!}=1-\frac{x}{2}\cot\frac{x}{2},\qquad |x|<2\pi.$$

The proof is accomplished by estimating the sum $\sum_{r=1}^{n}\cot^{2k}(r\pi/2n+1)$, for large n, in two different ways (Lemmas 1 and 2).

LEMMA 1.

$$\lim_{n\to\infty}\frac{1}{(2n)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)=\frac{2^{2k-1}}{(2k)!}B_k,\qquad k=1, 2, 3, \cdots.$$

Proof. For $k=1, 2, 3, \cdots$, let

$$s_n(k)=\frac{1}{(2n)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right).$$

Now the numbers $\cot(r\pi/2n+1)$, $r=\pm1, \pm2, \cdots, \pm n$, are the $2n$ roots of $(z+i)^{2n+1}-(z-i)^{2n+1}=0$. This equation can be written

$(\tau\sqrt{5}, \tau, \tau^{-2})$, $(2\tau, 2, 2\tau^{-1})$, $(\tau^2, \tau^2, \tau^{-1}\sqrt{5})$, $(\sqrt{5}, \sqrt{5}, \sqrt{5})$. Symmetry is that of the cube.

6c. Edge-first of a $\{5, 3, 3\}$ of edge $2\tau^{-2}$, with $\alpha=3^{-1/2}\tau^{-3}$: $(\tau^2, \tau^{-2}, \alpha\tau^4)$, $(\tau^2, \tau^{-1}, \alpha\tau^2\sqrt{5})$, $(\tau^2, 1, \alpha)$, $(\sqrt{5}, \tau^{-1}, \alpha\tau^5)$, $(\sqrt{5}, 1, \alpha\tau^3\sqrt{5})$, $(\sqrt{5}, \tau, \alpha\tau)$, $(2, 0, 2\alpha\tau^4)$, $(2, 1, 3\alpha\tau^3)$, $(2, \tau, \alpha\tau^4)$, $(2, 2, 0)$, $(\tau, 0, 4\alpha\tau^3)$, $(\tau, \tau^{-1}, \alpha\tau^2(2\tau^2+1))$, $(\tau, \tau, \alpha(\tau^5+1))$, $(\tau, 2, 2\alpha\tau^3)$, $(\tau, \sqrt{5}, \alpha\tau^3)$, $(1, 0, \alpha(\tau^6+1))$, $(1, 1, \alpha\tau^6)$, $(1, \tau, \alpha\tau^4\sqrt{5})$, $(1, \sqrt{5}, \alpha\tau^3\sqrt{5})$, $(1, \tau^2, \alpha\tau^2)$, $(\tau^{-1}, \tau^{-1}, \alpha\tau(\tau^5+1))$, $(\tau^{-1}, 2, 2\alpha\tau^4)$, $(\tau^{-1}, \tau^2, \alpha\tau^2\sqrt{5})$, $(0, \tau^{-2}, 3\alpha\tau^4)$, $(0, \sqrt{5}, 3\alpha\tau^3)$, $(0, \tau^2, 3\alpha\tau^2)$. Symmetry is that of the hexagonal prism.

6d. Face-first of a $\{5, 3, 3\}$ of edge $2\tau^{-2}$, with $\beta=5^{-1/4}\tau^{-5/2}$: $(\tau^2, 0, 2\beta\tau^2)$ $(\tau^2, \tau^{-1}, \beta\tau^3)$, $(\tau^2, 1, \beta\tau)$, $(\sqrt{5}, 0, 2\beta\tau^3)$, $(\sqrt{5}, 1, \beta\tau^4)$, $(\sqrt{5}, \tau, \beta\tau^2)$, $(2, \tau^{-1}, \beta\tau^3\sqrt{5})$, $(2, \tau, \beta\tau^2\sqrt{5})$, $(2, 2, 0)$, $(\tau, \tau^{-2}, \beta\tau^5)$, $(\tau, 1, \beta\tau^2(3\tau-1))$, $(\tau, \tau, 3\beta\tau^2)$, $(\tau, 2, 2\beta\tau^2)$, $(\tau, \sqrt{5}, \beta\tau)$, $(1, \tau^{-1}, 3\beta\tau^3)$, $(1, 1, \beta\tau^2(\tau+3))$, $(1, 2, 2\beta\tau^3)$, $(1, \sqrt{5}, \beta\tau^4)$, $(1, \tau^2, \beta)$, $(\tau^{-1}, 0, 2\beta\tau^4)$, $(\tau^{-1}, \tau, \beta\tau^5)$, $(\tau^{-1}, \tau^2, \beta\tau^3)$. Symmetry is that of the decagonal prism.

Reference

1. H. S. M. Coxeter, Regular Polytopes, 2nd ed., Macmillan, New York, 1963.

ON $\sum_{n=1}^{\infty} (1/n^{2k})$

KENNETH S. WILLIAMS, Carleton University, Ottawa

In this note we give a simple proof of the well-known result ([1], [3])

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1}\pi^{2k}B_k}{(2k)!}, \qquad k = 1, 2, 3, \cdots,$$

where B_k is the kth Bernoulli number, defined by

$$\sum_{k=1}^{\infty} B_k \frac{x^{2k}}{2k!} = 1 - \frac{x}{2}\cot\frac{x}{2}, \qquad |x| < 2\pi.$$

The proof is accomplished by estimating the sum $\sum_{r=1}^{n} \cot^{2k}(r\pi/2n+1)$, for large n, in two different ways (Lemmas 1 and 2).

LEMMA 1.

$$\lim_{n\to\infty} \frac{1}{(2n)^{2k}} \sum_{r=1}^{n} \cot^{2k}\left(\frac{r\pi}{2n+1}\right) = \frac{2^{2k-1}}{(2k)!} B_k, \qquad k = 1, 2, 3, \cdots.$$

Proof. For $k=1, 2, 3, \cdots$, let

$$s_n(k) = \frac{1}{(2n)^{2k}} \sum_{r=1}^{n} \cot^{2k}\left(\frac{r\pi}{2n+1}\right).$$

Now the numbers $\cot(r\pi/2n+1)$, $r=\pm 1, \pm 2, \cdots, \pm n$, are the $2n$ roots of $(z+i)^{2n+1}-(z-i)^{2n+1}=0$. This equation can be written

$$(1)\qquad \binom{2n+1}{1}z^{2n} - \binom{2n+1}{3}z^{2n-2} + \cdots + (-1)^n\binom{2n+1}{2n+1} = 0.$$

We note that $2(2n)^{2k}s_n(k)$ is the sum of the $2k$th powers of the roots of (1). Thus, by Newton's identity [2] for $n \geqq k$, we have on dividing through by $2(2n)^{2k}\binom{2n+1}{1}$

$$(2)\qquad s_n(k) - \frac{\binom{2n+1}{3}}{\binom{2n+1}{1}(2n)^2}s_n(k-1) + \cdots + (-1)^{k-1}\frac{\binom{2n+1}{2k-1}}{\binom{2n+1}{1}(2n)^{2k-2}}s_n(1) + (-1)^k\frac{\binom{2n+1}{2k+1}k}{\binom{2n+1}{1}(2n)^{2k}} = 0.$$

Next we take $k=1, 2, 3, \cdots$, successively in (2). As

$$\lim_{n\to\infty}\frac{\binom{2n+1}{2r+1}}{\binom{2n+1}{1}(2n)^{2r}} = \frac{1}{(2r+1)!}$$

for $r=1, 2, \cdots, k$, we see that $\lim_{n\to\infty} s_n(k)$ (exists) $=d_k$ (say), where $d_k(k=1, 2, 3, \cdots)$ is given recursively by

$$(3)\qquad d_k - \frac{d_{k-1}}{3!} + \cdots + (-1)^{k-1}\frac{d_1}{(2k-1)!} = (-1)^{k-1}\frac{k}{(2k+1)!}\,.$$

A simple inductive argument shows that $|d_k| < 1$, $k=1, 2, \cdots$, so that $\sum_{k=1}^{\infty} d_k x^{2k}$ converges absolutely for $|x|<1$. Thus using the product theorem for absolutely convergent series we have for $|x| < 1$

$$\begin{aligned}\left\{\sum_{k=1}^{\infty} d_k x^{2k}\right\}\sin x &= \left\{\sum_{k=1}^{\infty} d_k x^{2k}\right\}\left\{\sum_{l=0}^{\infty}\frac{(-1)^l x^{2l+1}}{(2l+1)!}\right\}\\ &= \sum_{m=0}^{\infty}\left\{\frac{d_m}{1!} - \frac{d_{m-1}}{3!} + \cdots + (-1)^{m-1}\frac{d_1}{(2m-1)!}\right\}x^{2m+1}\\ &= \sum_{m=0}^{\infty}(-1)^{m-1}\frac{m}{(2m+1)!}x^{2m+1} \text{ (using (3))}\\ &= \tfrac{1}{2}\{\sin x - x\cos x\}\end{aligned}$$

so that

$$\sum_{k=1}^{\infty} d_k x^{2k} = \frac{1}{2} - \frac{x}{2}\cot x = \frac{1}{2}\sum_{k=1}^{\infty} B_k 2^{2k}\frac{x^{2k}}{2k!}\,.$$

Equating coefficients we have $d_k=(2^{2k-1}/(2k)!)B_k$, which proves the result.

LEMMA 2.

$$\lim_{n\to\infty}\frac{1}{(2n+1)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)=\frac{1}{\pi^{2k}}\sum_{r=1}^{\infty}\frac{1}{r^{2k}},\qquad k=1,2,\cdots.$$

Proof. The function $\cot^{2k} z$ has a pole of order $2k$ at $z=0$ and is analytic in the annulus $0<|z|<\pi$. By Laurent's theorem there exist complex numbers $a_{-2k}\neq 0,\ a_{-(2k-1)},\ \cdots,\ a_{-1},\ a_0,\ a_1,\ \cdots$ such that

$$\cot^{2k} z=\frac{a_{-2k}}{z^{2k}}+\cdots+\frac{a_{-1}}{z}+a_0+a_1z+\cdots,$$

valid for $0<|z|<\pi$. Clearly $a_{-2k}=\lim_{z\to 0} z^{2k}\cot^{2k} z=(\lim_{z\to 0} z\cot z)^{2k}=1$. Let $a(z)=a_0+a_1z+a_2z^2+\cdots$ so that $a(z)$ is analytic in $|z|<\pi$. Thus in particular $a(z)$ is continuous on the compact set $\{z\mid |z|\leqq\pi/2\}$ and so is bounded there, that is, there is a real number $A(k)\geqq 0$ such that $|a(z)|\leqq A(k)$, for $|z|\leqq\pi/2$. But

$$a(z)=\begin{cases}\cot^{2k} z-\dfrac{a_{-2k}}{z^{2k}}-\cdots-\dfrac{a_{-1}}{z}, & 0<|z|\leqq\pi/2,\\ a_0, \quad z=0,\end{cases}$$

so that

$$\left|\cot^{2k} z-\frac{a_{-2k}}{z^{2k}}-\cdots-\frac{a_{-1}}{z}\right|\leqq A(k),\qquad 0<|z|\leqq\pi/2.$$

Hence there exists a real number $B(k)\geqq 0$ such that

$$\left|\cot^{2k} z-\frac{1}{z^{2k}}\right|\leqq A(k)+\frac{B(k)}{|z|^{2k-1}},\qquad 0<|z|\leqq\pi/2.$$

Taking $z=r\pi/2n+1$ $(r=1,2,\cdots,n)$ we have

$$\left|\cot^{2k}\left(\frac{r\pi}{2n+1}\right)-\frac{(2n+1)^{2k}}{\pi^{2k}r^{2k}}\right|\leqq A+\frac{B(k)(2n+1)^{2k-1}}{\pi^{2k-1}r^{2k-1}},$$

so that

$$\begin{aligned}\left|\frac{1}{2n+1)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)-\frac{1}{\pi^{2k}}\sum_{r=1}^{n}\frac{1}{r^{2k}}\right| &\leqq \sum_{r=1}^{n}\left|\frac{1}{(2n+1)^{2k}}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)\right.\\ \left.-\frac{1}{\pi^{2k}r^{2k}}\right| &\leqq \sum_{r=1}^{n}\left\{\frac{A(k)}{(2n+1)^{2k}}+\frac{B(k)}{\pi^{2k-1}(2n+1)r^{2k-1}}\right\}\\ &\leqq \frac{A(k)n}{(2n+1)^{2k}}+\frac{B(k)}{\pi}\,\frac{(1+\log n)}{(2n+1)},\end{aligned}$$

as

$$\sum_{r=1}^{n}\frac{1}{r^{2k-1}}\leqq\sum_{r=1}^{n}\frac{1}{r}<1+\int_1^n\frac{dt}{t}=1+\log n.$$

Hence as

$$\lim_{n\to\infty}\left\{\frac{A(k)n}{(2n+1)^{2k}}+\frac{B(k)}{\pi}\frac{(1+\log n)}{2n+1}\right\}=0$$

we have

$$\lim_{n\to\infty}\frac{1}{(2n+1)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)=\frac{1}{\pi^{2k}}\sum_{r=1}^{\infty}\frac{1}{r^{2k}}\cdot$$

THEOREM. $\sum_{n=1}^{\infty} 1/n^{2k}=(2^{2k-1}\pi^{2k}B_k)/(2k)!$, $k=1, 2, 3, \cdots$.

Proof. This follows immediately from Lemmas 1 and 2 as $\lim_{n\to\infty}(2n/2n+1)^{2k}=1$, for fixed k.

References

1. T. J. I' A. Bromwich, An Introduction to the Theory of Infinite Series, Macmillan, London, 1959, p. 298.
2. I. N. Herstein, Topics in Algebra, Blaisdell, New York, 1965, p. 208.
3. K. Knopp, Theory and Application of Infinite Series, Blackie, London, 1951, p. 237.

PRODUCTS OF TRIANGULAR MATRICES

NATIONAL SCIENCE FOUNDATION PROGRAM FOR HIGH SCHOOL STUDENTS
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The following theorem was given in [1] (with an interesting application), and given another proof in [2]. In the summer of 1971 some members of the above-named program gave the proof shown here, and the extension given below.

THEOREM. *Let $S_1, S_2, \cdots, S_n$ be $n\times n$ upper triangular matrices over a ring R such that the (i, i) entry of S_i is 0, then $S_1S_2\cdots S_n=0$.*

For $P=(x_1, x_2, \cdots, x_n)\in R^n$ we have $PS_1=(0, y_2, y_3, \cdots, y_n)$, $PS_1S_2=(0, 0, z_3, z_4, \cdots, z_n), \cdots, PM=0$ where $M=S_1S_2\cdots S_n$. Thus M is the zero map. That $M=0$ follows from taking P to be successively $(1, 0, 0, \cdots)$, $(0, 1, 0, \cdots), \cdots$. In case R has no identity we may adjoin an identity, obtaining a ring R' and apply the above argument to R' instead of R.

EXTENSION. The full force of the hypotheses was not used. For example, it is enough to assume about S_1 that its first column is zero, about S_2 that its first two columns are zero after the first term, and so on.

References

1. G. P. Barker, Triangular matrices and the Cayley-Hamilton theorem, this MAGAZINE, 44 (1971) 34–36.
2. M. Stojakovic, On a theorem of G. P. Barker on triangular matrices, this MAGAZINE, 44 (1971) 133–134.

Hence as

$$\lim_{n\to\infty}\left\{\frac{A(k)n}{(2n+1)^{2k}}+\frac{B(k)}{\pi}\,\frac{(1+\log n)}{2n+1}\right\}=0$$

we have

$$\lim_{n\to\infty}\frac{1}{(2n+1)^{2k}}\sum_{r=1}^{n}\cot^{2k}\left(\frac{r\pi}{2n+1}\right)=\frac{1}{\pi^{2k}}\sum_{r=1}^{\infty}\frac{1}{r^{2k}}\cdot$$

THEOREM. $\sum_{n=1}^{\infty} 1/n^{2k}=(2^{2k-1}\pi^{2k}B_k)/(2k)!$, $k=1, 2, 3, \cdots$.

Proof. This follows immediately from Lemmas 1 and 2 as $\lim_{n\to\infty}(2n/2n+1)^{2k}=1$, for fixed k.

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ON A PROBLEM IN NUMBER THEORY

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This note is concerned with the following,

THEOREM. *For every natural number k, there exists a natural number $f(k)$ such that all natural numbers n strictly greater than $f(k)$ have the following property: there are at least k composite natural numbers smaller than n and relatively prime to it.*

Terminology. The sequence of natural numbers begins with 1; 2 is the first prime, but 1 is not a composite number.

From Bertrand's postulate that for every $n \geqq 1$ there is a prime P such that $n < P \leqq 2n$, it follows that $P_{n+1} < 2P_n$ for all n, where P_n denotes the nth prime. From Nagura's result [2] that for every $n \geqq 25$ there is a prime P such that $n < P \leqq 6n/5$, it follows that $P_{n+2} < 2P_n$ for $n \geqq 4$. These facts will be used in the proofs of the two lemmas which will be presented before the proof of the theorem itself.

LEMMA 1. *For every natural number k, there exists a natural number $N(k)$ such that*

$$P_1 \cdot P_2 \cdot \ \ldots \ \cdot P_{N(k)} > P^2_{N(k)+k}. \tag{1}$$

Proof. We will use induction on k. For $k=1$, let $N(1)=4$ $(2\cdot 3\cdot 5\cdot 7>121)$. Suppose (1) is true for k. Then $N(k) \geqq 4$ and

$$\begin{aligned} P_1 \cdot P_2 \cdot \ \ldots \ \cdot P_{N(k)} \cdot P_{N(k)+1} &> P^2_{N(k)+k} \cdot P_{N(k)+1} > (P_{N(k)+k+2}/2)^2 \cdot P_{N(k)+1} \\ &> P^2_{N(k)+1+k+1}, \quad \text{since } P_{N(k)+1} > 4. \end{aligned}$$

Thus (1) is true for $k+1$ with $N(k+1)=N(k)+1$. We have also established that we can choose $N(k)$ so that $4 \leqq N(k) \leqq 3+k$.

LEMMA 2. *With the notation of Lemma* 1, *we have*

$$P_1 \cdot P_2 \cdot \ \ldots \ \cdot P_n > P^2_{n+k} \tag{2}$$

for every $n \geqq N(k)$.

Proof. We consider (2) for a fixed k; again, we will use induction—this time on n. For $n=N(k)$, (2) is true by Lemma 1. Now suppose it holds for n. Then

$$P_1 \cdot P_2 \cdot \ \ldots \ \cdot P_n \cdot P_{n+1} > P^2_{n+k} \cdot P_{n+1} > (P_{n+1+k}/2)^2 \cdot P_{n+1} > P^2_{n+1+k},$$

as required.

Proof of the Theorem. Let $f(k)=P^2_{N(k)+k-1}$. If $n>f(k)$, then $P^2_{N(k)+k-1+a} < n \leqq P^2_{N(k)+k+a}$ for some $a \geqq 0$. Consider the composite natural numbers P^2_1, $P^2_2, \cdots, P^2_{N(k)+k-1+a}$, all smaller than n. If there are no more than $k-1$ composite natural numbers less than and relatively prime to n, then at least $N(k)+a$

primes must divide n. That is,

$$n > P_1 \cdot P_2 \cdot \ \ldots \ \cdot P_{N(k)+a} > P^2_{N(k)+a+k},$$

by Lemma 2. Having reached a contradiction, we conclude that there are at least k natural numbers smaller than and relatively prime to n for all $n > f(k)$.

In view of the above result, we are confronted with another question. For a fixed k, what is the value of $g(k)$, the smallest natural number which can take the part of $f(k)$? A similar problem arises for $M(k)$, the smallest $N(k)$. We have $M(1)=4$, $M(2)=4$, $M(3)=M(4)=M(5)=M(6)=5$. As for $g(k)$, it is to be noted that even $P^2_{M(k)+k-1}$ is not $g(k)$. For example, $P^2_{M(1)}=49$, $P^2_{M(2)+1}=121$, $P^2_{M(3)+2}=289$, $P^2_{M(4)+3}=361$, $P^2_{M(5)+4}=529$, but $g(1)=30$ ([1]), $g(2)=60$, $g(3)=90$, $g(4)=90$, $g(5)=120$.

References

1. A. A. Fraenkel, Introduction to Mathematics, vol. 1, Massada, Tel Aviv, 1954, p. 57 (in Hebrew).
2. J. Nagura, On the interval containing at least one prime number, Proc. Japan Acad., 28 (1952) 177–181.

INTEGERS AND THE SUM OF THE FACTORIALS OF THEIR DIGITS

GEORGE D. POOLE, Texas Tech University

An interesting theorem appeared in this MAGAZINE [1] which stated the following:

For each positive integer n with m digits there exists an integer N such that the first m digits of N! are equal to n.

A result which is somewhat similar in nature and just as amusing is the fact that there is only a finite number of positive integers with the property that each one is equal to the sum of the factorials of its digits. For example,

$$145 = 1! + 4! + 5!$$

The following proof is also interesting in that the positive integers are separated into three groups and a different technique or observation is used on each group to establish the above fact.

THEOREM. *The only positive integers having the property that each one is equal to the sum of the factorials of its digits are* 1, 2, 145 *and* 40,585.

Proof. A computer can be used to establish that the above integers are the only integers between 1 and 2,000,000 (inclusively) having the required property (a program can be furnished to the interested reader).

For the integers between 2,000,000 and 2,999,999 we make the following observations. Since the associated sum (the sum of the factorials of its digits) of any integer $2{,}000{,}000 \leqq N \leqq 2{,}899{,}999$ lies between 8 and 1,854,722 ($0!=1$ of primes must divide n. That is,

$$n > P_1 \cdot P_2 \cdot \ \ldots \ \cdot P_{N(k)+a} > P^2_{N(k)+a+k},$$

by Lemma 2. Having reached a contradiction, we conclude that there are at least k natural numbers smaller than and relatively prime to n for all $n > f(k)$.

In view of the above result, we are confronted with another question. For a fixed k, what is the value of $g(k)$, the smallest natural number which can take the part of $f(k)$? A similar problem arises for $M(k)$, the smallest $N(k)$. We have $M(1)=4$, $M(2)=4$, $M(3)=M(4)=M(5)=M(6)=5$. As for $g(k)$, it is to be noted that even $P^2_{M(k)+k-1}$ is not $g(k)$. For example, $P^2_{M(1)}=49$, $P^2_{M(2)+1}=121$, $P^2_{M(3)+2}=289$, $P^2_{M(4)+3}=361$, $P^2_{M(5)+4}=529$, but $g(1)=30$ ([1]), $g(2)=60$, $g(3)=90$, $g(4)=90$, $g(5)=120$.

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Similarly, the associated sum of any integer $2{,}900{,}000 \leqq N \leqq 2{,}999{,}999$ lies between 362,887 and 2,177,282 and, hence, cannot have the required property.

We now consider the group of integers $N \geqq 3{,}000{,}000$ having an i number of digits. Consider the "new" associated sum of each member of $N(i)$ to be $i \cdot 9!$. Since its actual associated sum can be no greater than $i \cdot 9!$, all we need do is show that each element of $N(i)$ is greater than $i \cdot 9!$ and, therefore, greater than its actual associated sum; consequently never equal. It is sufficient to verify this fact for the smallest element in each set $N(i)$. This we show by induction on i.

For $i=7$ we observe that 3,000,000 is smallest in $N(7)$ and $3{,}000{,}000 > 7 \cdot 9! = 2{,}540{,}160$. Similarly, for $i=8$ we have $10{,}000{,}000 > 8 \cdot 9! = 2{,}903{,}040$.

Assume when $i=k \geqq 8$ that the smallest integer in $N(k)$ is larger than $k \cdot 9!$. That is, $10^{k-1} > k \cdot 9!$. To establish this fact for $i=k+1$ we observe that

$$10^k - 362{,}880 > 10^{k-1} > k \cdot 9!$$

and, therefore,

$$10^k > k \cdot 9! + 362{,}880 = (k+1) \cdot 9!$$

where $9! = 362{,}880$. This completes the proof.

It would be of some interest to know if a proof exists without making use of the "exhaustion" technique.

The author wishes to thank the referee for pointing out an important oversight.

Reference

1. J. M. Maxfield, A note on $N!$, this MAGAZINE, 43 (1970) 64–67.

CIRCLE THROUGH THREE GIVEN POINTS

MURRAY S. KLAMKIN, Ford Motor Company, Dearborn, Michigan

A usual exercise in analytic geometry is to determine the circle (equation, center and radius) passing through three specified points (x_i, y_i), $i=1, 2, 3$. In order to simplify the arithmetical calculations, one should choose the given points as lattice points such that the center is also a lattice point and that the radius is integral. A general way of doing this would be first to consider the center to be at the origin (since it can always be translated there). We then determine r, such that there are at least three distinct solutions for the Diophantine equation

$$x_i^2 + y_i^2 = r^2.$$

As is well known, the sides of an integral right triangle must be of the form

$$2mn, \quad m^2 - n^2, \quad m^2 + n^2.$$

Thus, $r = m^2 + n^2$. A rather trivial solution for the (x_i, y_i) will be permutations of course), then there is no integer N in this group having the required property.

Similarly, the associated sum of any integer $2{,}900{,}000 \leqq N \leqq 2{,}999{,}999$ lies between 362,887 and 2,177,282 and, hence, cannot have the required property.

We now consider the group of integers $N \geqq 3{,}000{,}000$ having an i number of digits. Consider the "new" associated sum of each member of $N(i)$ to be $i \cdot 9!$. Since its actual associated sum can be no greater than $i \cdot 9!$, all we need do is show that each element of $N(i)$ is greater than $i \cdot 9!$ and, therefore, greater than its actual associated sum; consequently never equal. It is sufficient to verify this fact for the smallest element in each set $N(i)$. This we show by induction on i.

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CIRCLE THROUGH THREE GIVEN POINTS

MURRAY S. KLAMKIN, Ford Motor Company, Dearborn, Michigan

A usual exercise in analytic geometry is to determine the circle (equation, center and radius) passing through three specified points (x_i, y_i), $i=1, 2, 3$. In order to simplify the arithmetical calculations, one should choose the given points as lattice points such that the center is also a lattice point and that the radius is integral. A general way of doing this would be first to consider the center to be at the origin (since it can always be translated there). We then determine r, such that there are at least three distinct solutions for the Diophantine equation

$$x_i^2 + y_i^2 = r^2.$$

As is well known, the sides of an integral right triangle must be of the form

$$2mn, \quad m^2 - n^2, \quad m^2 + n^2.$$

Thus, $r = m^2 + n^2$. A rather trivial solution for the (x_i, y_i) will be permutations of

$(\pm 2mn, \pm(m^2-n^2))$. Some of these correspond to the given three points forming right triangles. To obtain general type triangles, we have to select r such that there is more than one nontrivial representation of it by a sum of squares.

Gauss and Legendre, ([1], p. 227) showed that if

$$r = 2^u S \prod_i p_i^{\alpha_i},$$

where S is a product of all the prime factors of r of the form $4n+3$ and p_i are distinct primes of the form $4n+1$, then if S is a square there are

$$k = \frac{1}{2} \prod_i (\alpha_i + 1)$$

representations of r as a sum of two squares, when one of the exponents α_i is odd; but $k+1/2$ if the α_i are all even. If S is not a square, there are no representations. (Here the squares and not their roots are counted.) For example, $5^2 \cdot 13$ has three representations. These can be found by expressing each prime of the form $4n+1$ as a sum of squares which is unique and using

$$(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2.$$

Thus,

$$5^2 \cdot (3^2 + 2^2) = 15^2 + 10^2 = (12 \pm 6)^2 + (9 \mp 8)^2.$$

Aside from an arbitrary lattice translation, the latter special case leads to the following 8^3 combinations of points for the (x_i, y_i):

$$(x_1, y_1) \text{ or } (y_1, x_1) = (\pm 2 \cdot 15 \cdot 10 + h, \pm(15^2 - 10^2) + k),$$

$$(x_2, y_2) \text{ or } (y_2, x_2) = (\pm 2 \cdot 18 \cdot 1 + h, \pm(18^2 - 1^2) + k),$$

$$(x_3, y_3) \text{ or } (y_3, x_3) = (\pm 2 \cdot 17 \cdot 6 + h, \pm(17^2 - 6^2) + k).$$

We now return to determining the circle passing through three given arbitrary noncollinear points. Even this routine exercise can pose a challenge to the creativity of the better student if he is asked to solve the problem in as many different ways as he can. This pedagogical challenge can also be used on other routine exercises. For example, the author has found over 10 ways of finding the distance from a point to a line and of finding good approximations to $\sqrt{2}$.

If all we want is the equation, we can write it down immediately in determinant form:

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0.$$

However, for the rest of the determinations, this particular form then requires the evaluation of four 3rd order determinants (which can be simplified by the usual elementary operations).

Another form which can be written down immediately (and is believed to be new) is

$$\sum_{\text{cyclic}} \frac{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)}{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)} = 1. \tag{1}$$

Another method given by Bond [2], deserves to be more widely known. The equation for the family of circles passing through the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ is given by

$$C(P_1, P_2) + \lambda L(P_1, P_2) = 0,$$

where

$$C(P_1, P_2) \equiv (x-x_1)(x-x_2)+(y-y_1)(y-y_2),$$

$$L(P_1, P_2) \equiv (y_2-y_1)(x-x_1)-(x_2-x_1)(y-y_1),$$

and

λ is an arbitrary parameter.

Here $C=0$ is the equation of the circle with P_1P_2 as a diameter and $L=0$ is the equation of the line through P_1 and P_2. The value of λ is determined by requiring the equation to be satisfied by the third point $P_3(x_3, y_3)$.

In the previous three methods, we still will have to complete squares in order to determine the center and radius.

The fourth and last method, which is probably the most direct and the one most used, is to find the center first as the intersection of perpendicular bisectors of the sides of the triangle. This leads immediately to the two linear equations

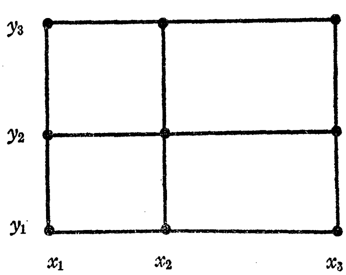
$$(x-x_1)^2+(y-y_1)^2 = (x-x_2)^2+(y-y_2)^2 = (x-x_3)^2+(y-y_3)^2$$

which are easily solved. Then the radius and finally the equation is determined.

In setting up (1), the author was also led to consider the equation

$$\sum_{\text{cyclic}} \left\{ \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} + \frac{(y-y_1)(y-y_2)}{(y_3-y_1)(y_3-y_2)} \right\} = 2.$$

It follows by inspection that the latter equation is satisfied by the nine points (x_i, y_j), $i, j=1, 2, 3$. Since the equation is ostensibly a quadratic, it is questionable how a conic section can pass through the nine points pictured in the figure.



This may be an interesting problem to pose to a class. Of course, the answer is that no conic section can pass through the pictured nine points, and consequently, the "equation" must be an identity. This follows by just considering the "x part." Since it is quadratic and takes on the value 1 for $x=x_1, x_2, x_3$ it must identically equal one.

One standard way of generalizing problems is to increase the dimensionality. This leads here to the problem of determining the sphere passing through four given noncoplanar points. We can simplify the arithmetic involved by determining four sets of integral solutions to

$$x_i^2 + y_i^2 + z_i^2 = r^2$$

for fixed r. One can do this by means of identities due to Euler and Catalan, ([1], p. 266):

$$(a^2 + b^2 + c^2)^2 = (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2,$$

$$(a^2 + b^2 + c^2 + ab + bc + ca)^2 = (a + c)^2(b + c)^2 + (b + c)^2(a + b)^2 + (c^2 + ac + bc - ab)^2.$$

Note that for the r.h.s's, we can cyclically interchange the letters a, b, c, insert minus signs in the parentheses and change the order of the terms. Additional representations can be gotten from the special case when

$$ab + bc + ca = 0,$$

which is equivalent to the optic formula

$$\frac{1}{-c} = \frac{1}{a} + \frac{1}{b}\cdot$$

The latter has solutions of the form ([1] p. 689)

$$-c = \lambda mn, \qquad a = \lambda m(m + n), \qquad b = \lambda n(m + n).$$

The direct way of actually determining the sphere through four given noncoplanar points is analogous to that for the circle. The center will be given by the intersection of perpendicular bisecting planes of three noncoplanar edges of the tetrahedron determined by the four points. These equations are

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = (x - x_4)^2 + (y - y_4)^2 + (z - z_4)^2, \qquad i = 1, 2, 3.$$

The rest follows easily.

Other methods, analogous to the other methods for the circle have been treated in a recent paper, [3].

References

1. L. E. Dickson, History of Theory of Numbers, vol. 2, Stechert-Hafner, New York 1934.
2. C. Bond, The equation of a circle, Math. Gaz., (1966) 154–155.
3. M. S. Klamkin, The equation of a sphere, this MAGAZINE, 42 (1969) 241–242.

BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to: Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital **C** *in the margin indicates a classroom review.*

C *Analysis* I. By Serge Lang. Addison Wesley, Reading, 1968. 460 pp. $12.50.

After (I) a review of elementary analysis (of functions in R into R), in which an axiomatic treatment of the integral as an interval-function is an interesting touch, the book gets down to business (II) with a concise treatment of normed vector spaces, of continuous mappings of one normed vector space into another, and of integrals as uniform limits of step-mappings (of an interval into a normed vector space).

A section (III) on Dirac sequences, convolutions, and Fourier series is rather a bypath, as the author indeed warns us, and the real meat of the book starts in part IV: calculus on (complete normed) vector spaces, using the Fréchet derivative. The fifth and last part is on (Riemann-Darboux) integration of functions in R^n into R, ending with a chapter on differential forms.

Although the selection of material in this text is attractive, its organization and detailed treatment are disappointing. The chapter on differential forms seems to lead nowhere; and I wonder whether it should have been the first chapter in volume II rather than the last in volume I. Integration of functions in R^n comes as somewhat of an anticlimax after differentiation in the more general setting of normed vector spaces. Chapter IX consists almost entirely of a review of series of real numbers and of series of functions in R into R and would fit better in Chapter I: true, the theorem about rearrangement of an absolutely convergent series is stated and proved for series of elements of a normed vector space, but the proof is the same as for series of real numbers. In any case, the theorem seems out of place in a text of this sort: the important series for analysis are power-series, and the last thing we ever want to do to a power series is t6 rearrange it.

The definition of such a fundamental concept as that of limit always needs care; at this level the definition needs particular care, because it is precisely now that many students will be passing from the traditional definition (L is a limit of f at a if $f(x)$ is arbitrarily close to L for all x close enough to a) to the modern one (L is a limit of f at a if a is adherent to the domain of f, and $f(x)$ is arbitrarily close to L for all x in the domain of f close enough to a). Lang's definition leaves the part played by the domain unclear:

Let S be a set of numbers, let a be adherent to S, and let f be a function defined on S. L is a limit of f at a if $f(x)$ is arbitrarily close to L for every x in S close enough to a.

The $64 question is: what does "f is defined on S" mean? Does it mean dom

$f=S$, as the definition on page 36 seems to say? Or does it mean dom $f \supseteq S$? If the former, then

$$\lim_{\substack{x \to a \\ x \in S}} f(x)$$

is not defined unless $S=\text{dom } f$; the limit (of g at 0 in T) on page 39 is not defined; and the remark " . . . our definition of limit depends on the set S on which f is defined" is incorrect, because S is determined by f. But if we take the second interpretation, then the statement (proposition 1, page 37, repeated on page 116) that if the limit exists, it is unique, is false: let $f(\xi)=|\xi|/\xi$ for every nonzero ξ, S be the set of all positive numbers, and T the set of all negative numbers; then the limits of f at 0 in S and in T both exist and are unequal.

This may be making a mountain out of a molehill (and indeed the faulty definition leads to no dire consequences); nevertheless this kind of argument is the very stuff of analysis and no student who made a comparable mistake in an exercise would have it marked correct.

The course for which we used the book was taught in several sections, and my colleagues seem in general agreement that the book is in the general spirit of modern mathematics, but falls short in too many details for them to be willing to recommend its use again.

For these reasons, and because the students found the book's arguments substantially harder to follow than those in other similar books, I would regard this book as more suitable as background reading than as a text for a course of lectures.

HUGH THURSTON, University of British Columbia

Introduction to Real Analysis. By Michael Gemignani. W. B. Saunders, Philadelphia, 1971. 160 pp.

Texts intermediate between elementary and advanced calculus have long been needed. Recently several of them have come on the market. The table of contents of this book puts it into the intermediate calculus category; so do the stipulated prerequisites of two semesters of a standard calculus course.

The book is intended for a one semester course and includes the central topics. After some preliminaries about sets, relations, and cardinality, chapter 2 gives a construction of the reals by means of Dedekind cuts, starting from an ordered integral domain. Basic concepts of real line topology are discussed in chapter 3: completeness, compactness, connectedness; continuity and uniform continuity. These first three chapters lay the foundation for the calculus. The remaining four chapters are concerned, respectively, with sequences and series, the derivative, the Riemann integral, and sequences and series of functions. The coverage of theorems appears adequate and the notation modern.

If pressed for time an instructor could shorten chapters 4, 6, and 7 insofar as the material is presented in advanced calculus texts in very nearly the same form. If the book is used in a terminal course, however, one would be less willing to leave out any of the material and would perhaps even want to supplement it

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Will the book achieve the author's aim " . . . to present in a well-motivated and natural sequence the basic ideas of classical real analysis"? The sequence of ideas is natural enough but it suffers from discreteness and disconnectedness (as sequences are apt to do). Almost entirely lacking is the occasional remark designed to arouse the student's interest, to fire his imagination, to reassure him that he will be able to jump the next hurdle, to put things into perspective. It is easy to enumerate ideas, but what is their essential nature? Why do they "work"? How do they enter into recurrent patterns? How, why and by whom were they invented in the first place? A well-motivated (or motivating) book must pose and answer such questions. In this book one finds the occasional "this theorem is very important in mathematics"; that is surely not sufficient motivation.

The style of the presentation is straightforward, if a bit terse. Statements of definitions and theorems seem clear and complete. Proofs are standard, that is, the average student will need help with them. There are quite a few good examples given.

The mathematical level of the book is perhaps a bit higher than that for which an average group of mathematics majors is prepared at the end of their first year of calculus. Basically the material should be reachable by such students, yet it comes thick and fast. The exercises are mostly of the "prove that . . ." type. The student who meets mathematical ideas (in rigor) for the first time will need a lot of problems of the type "verify that . . . is a . . ." and "find examples and counterexamples for . . ."—simple, encouraging exercises, some of them worked through in detail.

Is there time enough for all this in a one semester course? The answer will depend largely on the composition of the individual class, and so will the usability of this book as a text.

A. BAARTZ, University of Victoria

THE GREATER METROPOLITAN NEW YORK MATH FAIR

The fourth annual Greater Metropolitan New York Math Fair will be held on Sunday, March 5, 1972 at Pace College in New York City.

The purpose of the Fair is to foster research and interest in mathematics among high school students and to enable them to pursue independently some phase of mathematics of their choice by presenting papers on their topics to a group of judges.

Applications, accepted from high school students studying 11th Year Mathematics or higher, should he submitted by December 10, 1971 to:

Professor Theresa J. Barz, Secretary
Math Fair Committee
Dept. of Mathematics, St. John's University
Jamaica, New York 11432

The papers, due February 9, 1972, need not represent original mathematics with a discussion of Lebesgue's necessary and sufficient condition for Riemann integrability.

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Judges are urgently needed to ensure the success of the Fair. Mathematics teachers and professors interested in performing this service on March 5 or at the March 12 final round please contact Professor Barz.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles Valley College

ASSOCIATE EDITOR, J. S. FRAME, Michigan State University

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Problems may be submitted from any branch of mathematics and ranging in subject content from that accessible to the talented high school student to problems challenging to the professional mathematician. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

The asterisk (*) will be placed by the problem number to indicate that the proposer did not supply a solution. Readers' solutions are solicited for all problems proposed. Proposers' solutions may not be "best possible" and solutions by others will be given preference.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles Valley College, 5800 Fulton Avenue, Van Nuys, California 91401.

To be considered for publication, solutions should be mailed before May 15, 1972.

PROBLEMS

810. *Proposed by Zalman Usiskin, University of Michigan.*

Solve the following cryptarithm (in base 10):

$$\begin{array}{rcccccc} & & & & S & I & X \\ \times & & & & T & W & O \\ \hline & & & - & - & - & - \\ & & - & - & - & - & \\ & - & - & - & - & & \\ \hline T & W & E & L & V & E & \end{array}$$

811. *Proposed by James Bookey, Mount Senario College, Wisconsin.*

Find a domain bounded by a simple closed polygon such that for each two sides of the polygon there is an interior point from which these two sides are visible, but such that there is no interior point from which all sides are visible.

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812. *Proposed by Donald P. Minassian, Butler University, Indiana.*

Two fully ordered groups G and H are order-isomorphic if there is an isomorphism f from G to H which preserves orderings: $a>b$ in G if and only if $f(a)>f(b)$ in H. Let R be the additive group of real numbers under the usual ordering. Show that no proper extension, and no proper subgroup of R is order-isomorphic to R: we assume that the extension and subgroup are ordered to preserve the ordering of R.

813. *Proposed by L. Carlitz and R. A. Scoville, Duke University.*

Show that any polynomial with real coefficients can be written as a difference of two real monotone increasing polynomials.

814. *Proposed by Marlow Sholander, Case Western Reserve University.*

For what values of a is the graph of a^x tangent to the graph of $\log_a x$?

815. *Proposed by Sidney H. L. Kung, Jacksonville University, Florida.*

Given two complex numbers z_1 and z_2 whose sum is z. Let the angle between the vectors $0z_1$ and $0z$ be designated by θ. Let $0z_3$ and z_1z_3 intersect at z_3 such that:

(a) angle $z_2 0 z_3=\theta=$ angle $0z_1z_3$,

(b) the vector $0z_3$ lies within the angle ϕ subtended by $0z_1$ and $0z_2$; $0<\phi<\pi$.

Prove that $z_3=(z_1^{-1}+z_2^{-1})^{-1}$.

816. *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Show that no equilateral triangle which is either inscribed in or circumscribed about a noncircular ellipse can have its centroid coincide with the center of the ellipse.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q528. If $781\times 965=753,A65$, without multiplying find A.

[*Submitted by P. G. Pantelidakis*]

Q529. Show that $\int_0^1 [1-(1-t)^n]t^{-1}dt=1+1/2+\cdots+1/n$.

[*Submitted by Frank J. Papp*]

Q530. Suppose $a-1$ and $a+1$ are twin primes larger than 10. Prove that a^3-4a is divisible by 120.

[*Submitted by George E. Andrews*]

Q531. There are many closed subsets of a solid torus which, like the meridian sections, intercept each longitudinal circle in the torus exactly once. Are there any which also contain the points of a simple (closed) spiral on the surface?

[*Submitted by R. A. Struble*]

(Answers on pages 298–299.)

SOLUTIONS

Nested Simplexes

782. [January, 1971] *Proposed by Herta T. Freitag, Hollins, Virginia.*

1. Into an n-dimensional regular simplex of side a an n-dimensional hypersphere is inscribed. Inscribe another regular simplex into this hypersphere. Continue in this manner *ad infinitum.* Find:

a. the total contents of the entire set of simplexes.

b. the total contents of the entire set of hyperspheres.

2. Deal with the analogous problem concerning an n-dimensional hypercube.

Solution by Michael Goldberg, Washington, D.C.

(1) The radius r of an inscribed hypersphere of n-dimensional regular simplex of edge a is given by the equation

$$r = \{1/2n(n+1)\}^{1/2}a.$$

(See *Regular Polytopes*, H. S. M. Coxeter, in table of page 295).

The radius R of the circumsphere is nr. The edge a_1 of the simplex in the inscribed hypersphere is given by the equation

$$a_1 n\{1/2n(n+1)\}^{1/2} = r = \{1/2n(n+1)\}^{1/2}a.$$

Hence, $a_1 = a/n$. Similarly, $a_2 = a/n^2$, $a_3 = a/n^3$, etc.

The content C of a regular simplex of edge a is given by the equation

$$C = \frac{a^n}{n!}\left(\frac{n+1}{2^n}\right)^{1/2}.$$

Hence, the total contents of the entire set of simplexes is given by the equation

$$\frac{1}{n!}\left(\frac{n+1}{2^n}\right)^{1/2} \sum a_k^n = \frac{a^n}{n!}\left(\frac{n+1}{2^n}\right)^{1/2} \sum_{k=0}^{\infty} 1/n^k$$
$$= \frac{a^n}{n!}\left(\frac{n+1}{2^n}\right)^{1/2}\left(\frac{n}{n-1}\right).$$

The content S_n of a hypersphere of radius R in n dimensions is given by the equation $S_n = 2\pi^{n/2}R^n/\Gamma(n/2)$. (Coxeter, pp. 125–126.) Hence, the sum of the contents of the hyperspheres is given by

$$\frac{2\pi^{n/2}}{(n/2)}\{1/2n(n+1)\}^{n/2}a^n \sum_{k=0}^{\infty} 1/n^k = \frac{2\pi^{n/2}}{\Gamma(n/2)}\{1/2n(n-1)\}^{n/2}a^n\left(\frac{n}{n-1}\right).$$

(2) For hypercubes, the ratios of the lengths of the successive edges is $1/\sqrt{n}$. Hence, the sum of the contents of the hypercubes is

$$\sum_{k=0}^{\infty} 1/n^{k/2} = \sqrt{n}/(\sqrt{n}-1).$$

Similarly, the sum of the contents of the hyperspheres is

$$\frac{2\pi^{n/2}(a/2)^n}{\Gamma(n/2)}\left(\frac{\sqrt{n}}{\sqrt{n}-1}\right).$$

Also solved by Heiko Harborth, Braunschweig, Germany and the proposer.

Saalschützian Series

783. [January, 1971] *Proposed by H. L. Krall, University Park, Pennsylvania.*

1. If a_{ij} is the binomial coefficient $\binom{p}{q-i+j}$, evaluate d_n where

$$d_n = \begin{vmatrix} a_{11} & \cdots & a_{1^n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{n_1} & \cdots & a_{nn} \end{vmatrix}.$$

2. Show that $\sum_{k=0}^{n}\binom{n}{k}x_{\underline{k}|}y_{\underline{k}|}z_{\underline{n-k}|}(x+y+z-k)_{\underline{n-k}|} = (x+z)_{\underline{n}|}(y+z)_{\underline{n}|}$ where

$$x_{\underline{n}|} = x(x-1)\cdots(x-n+1).$$

Solution by H. W. Gould, West Virginia University.

By adding rows successively in the given determinant and using the recurrence relation $\binom{x}{i-1}+\binom{x}{i}=\binom{x+1}{i}$, it is easy to see that the given nth order determinant with general element $a_{ij}=\binom{p}{q-i+j}$ has the same value as the nth order determinant with general element $a_{ij}=\binom{p+i-1}{q+j-1}$. Recalling that $\binom{x}{i}=\frac{x}{i}\binom{x-1}{i-1}$ and using this repeatedly, we see that the determinant factors, and indeed, p is a factor of every element in the first row, $1/q$ is a factor of every element in the first column, $p+1$ is a factor of every element in the second row, $1/(q+1)$ is a factor of every element in the second column, etc. Then the ratio $(p-1)/(q-1)$ is seen to be a factor, $(p-2)/(q-2)$, etc., and carrying this factorization to its conclusion we obtain finally the desired formula for the value of the determinant:

$$D_n = \prod_{i=0}^{n-1} \frac{\binom{p+i}{q+1}}{\binom{p-q+i}{i}}.$$

The determinant seems first to have been evaluated by V. Zeipel in 1865. Details of his work and other valuable references may be found in the book *Die Determinanten*, by Ernesto Pascal, Teubner, Leipzig, 1900, pp. 133–134. The method above is that given by Pascal. Pascal gives other valuable references in the older literature dealing with determinants whose elements are binomial coefficients.

The formula to be proved may be stated in the equivalent form

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} \binom{x}{k}\binom{y}{k}\binom{z}{n-k}\binom{x+y+z-k}{n-k} = \binom{x+z}{n}\binom{y+z}{n}, \tag{1}$$

and the formula will be shown to be true for all real or complex x, y, z, and integers $n \geqq 0$. There are several ways to prove the formula using formulas for hypergeometric functions; however, our proof below will use just the Vandermonde convolution (addition theorem)

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}, \tag{2}$$

known to be true for all real or complex x and y, and some operations on finite series. We shall first prove (1) in the special case that x is a nonnegative integer and then show that the extension to arbitrary x follows at once. To be precise, (1) is a polynomial identity in x, y, z, and so if (1) can be established for more values of x than the degree of the polynomial in x, it is easy to see that the formula must then be true for all real x.

Now it is easy to see that

$$\binom{x}{k}\binom{z}{n-k}\binom{n}{k}^{-1} = \binom{x+z-n}{x-k}\binom{x+z}{n}\binom{x+z}{x}^{-1} \tag{3}$$

when x is a nonnegative integer. Using this, it is clear that (1) is equivalent to the formula

$$\sum_{k=0}^{n} \binom{y}{k}\binom{x+y+z-k}{n-k}\binom{x+z-n}{x-k} = \binom{x+z}{x}\binom{y+z}{n} \tag{4}$$

which we will prove valid for nonnegative integers x, n and real y, z. In fact

$$\begin{aligned}
&\sum_{k=0}^{n} \binom{y}{k}\binom{x+y+z-k}{n-k}\binom{x+z-n}{x-k} \\
&\qquad = \sum_{k=0}^{n} \binom{y}{k}\binom{x+z-n}{x-k}\sum_{j=0}^{n-k}\binom{x+z}{j}\binom{y-k}{n-k-j}, \qquad \text{by (2)} \\
&\qquad = \sum_{j=0}^{n} \binom{x+z}{j}\sum_{k=0}^{n-j}\binom{y}{k}\binom{x+z-n}{x-k}\binom{y-k}{n-j-k}, \\
&\qquad = \sum_{j=0}^{n} \binom{x+z}{n-j}\sum_{k=0}^{j}\binom{y}{k}\binom{x+z-n}{x-k}\binom{y-k}{j-k}, \\
&\qquad = \sum_{j=0}^{n} \binom{x+z}{n-j}\binom{y}{j}\sum_{k=0}^{j}\binom{j}{k}\binom{x+z-n}{x-k},
\end{aligned}$$

$$= \sum_{j=0}^{n} \binom{x+z}{n-j}\binom{y}{j}\binom{x+z-n+j}{x}, \qquad \text{by (2)}$$

$$= \binom{x+z}{x}\sum_{j=0}^{n}\binom{y}{j}\binom{z}{n-j},$$

$$= \binom{x+z}{x}\binom{y+z}{n}, \qquad \text{by (2).}$$

The proof of (4) is then complete. Converting this back into the form (1) of a polynomial identity in x, y, z, the general result follows.

We remark finally that (1) may be cast in the equivalent form

$$(5) \qquad \sum_{k=0}^{n} \frac{\binom{x}{k}\binom{y}{k}\binom{z}{n-k}}{\binom{x+y+z}{k}} = \frac{\binom{x+z}{n}\binom{y+z}{n}}{\binom{x+y+z}{n}}$$

which is also valid for all real or complex x, y, z. This form is closely related to a form of Saalschutz.

Also solved by L. Carlitz, Duke University; Eldon Hansen, Lockheed Research Laboratory, Palo Alto, California; and the proposer.

A Ring Property

784. [January, 1971] *Proposed by Robert S. Doran, Texas Christian University.*

Let A be a ring with identity 1 without divisors of zero, and suppose $f: A \to A$ is a ring antihomomorphism such that $f(f(a)) = a$. Show that a and $f(a)$ commute whenever $af(-a) = 1$.

I. *Solution by Eugen Peter Bauhoff, Frankfurt, West Germany.*

Suppose that $a \cdot f(a) = 1$ and put $f(a) = x$. Then $f(-a) = -x$ and

$$(1) \qquad a \cdot (-x) = 1.$$

We have to show that $a \cdot x = x \cdot a$.

Multiplying (1) with a from the right, we get

$$(2) \qquad a \cdot (-x) \cdot a = a.$$

Therefore

$$(3) \qquad a \cdot (-x) \cdot a - a = a((-x) \cdot a - 1) = 0.$$

Since A contains no divisors of zero and since $a \neq 0$ by (1), we get

$$(4) \qquad (-x) \cdot a = 1.$$

From (1) and (4) it follows immediately that $a \cdot x = x \cdot a$.

The condition $f(f(a)) = a$ and the antihomomorphism-property of f have not been used in the proof.

II. *Solution by Bob Prielipp, Wisconsin State University at Oshkosh.*

We shall establish the following stronger result: Let A be a ring with identity 1 without divisors of zero, and suppose $f:A\rightarrow A$ such that $f(x+y) = f(x)+f(y)$ for each x, $y\in A$. Then a and $f(b)$ commute whenever $af(-b)=1$.

By hypothesis $f(x+y)=f(x)+f(y)$. Thus $f(0)=0$. Hence $0=f(0)=f(b+(-b)) = f(b)+f(-b)$ and $f(-b)=-f(b)$.

If $a=0$ then clearly $af(b)=f(b)a$. Thus in the remainder of our solution we shall assume that $a\neq 0$. By hypothesis $af(-b)=1$. Hence $a(f(-b)a)=(af(-b))a = 1\cdot a = a\cdot 1$. Since A has no divisors of zero, $f(-b)a=1$. Thus $af(-b)=f(-b)a$ or $a(-f(b))=(-f(b))a$. Therefore $-(af(b))=-(f(b)a)$, from which it follows immediately that $af(b)=f(b)a$.

Also solved by Arthur R. Bolder, Brooklyn, New York; Edmund M. Clarke, Madison College; George Corliss, Michigan State University; William F. Fox, Moberly Junior College, Missouri; Stephen I. Gendler, Clarion State College, Pennsylvania; M. G. Greening, University of New South Wales, Australia; Greg Jennings, University of Puget Sound, Washington; Richard Kerns, Oak Lawn Illinois; Henry S. Lieberman, Waban, Massachusetts; David E. Manes, State University College, Oneonta, New York; W. Margolis, Colgate University; Joseph V. Michalowicz, Catholic University of America; C. Bruce Myers, Austin Peay State University, Tennessee; Thomas O'Loughlin, SUNY, Cortland, New York; Robert B. Reisel, Loyola University of Chicago; Rina Rubenfeld, New York City Community College, Brooklyn, New York; E. F. Schmeichel, College of Wooster, Ohio; Edward C. Waymire, Southern Illinois University; Albert White, St. Bonaventure University, New York; E. T. Wong, Oberlin College, Ohio; and the proposer, who resubmitted the problem with the weakened hypothesis.

Triangles in a Square

785. [January, 1971] *Proposed by Michael Goldberg, Washington, D.C.*

In Problem 745, this MAGAZINE, solution May, 1970, we were asked "Given any nine points in a unit square, show that among the triangles having vertices on the given points there exists at least one triangle whose area does not exceed 1/8." Closer bounds can be found.

a. In particular, find an arrangement of ten points such that the smallest triangle has an area of $(3\sqrt{17}-11)/32=0.0428$.

b. Find an arrangement of nine points in which the smallest triangle has an area greater than in a but less than 1/8.

Solution by the proposer.

(a) Eight points are placed on the edges of the square, two on each side. Each point is at a distance of $(5-\sqrt{17})/4$ from a corner. Each of the two remaining points is placed on a diagonal of the square at a distance of $\sqrt{2}(9-\sqrt{17})/16$ from a corner. Then the area of the smallest triangle is $(3\sqrt{17}-11)/32=0.0428$.

(b) Consider the arrangement of the nine points which are shown as circles points labelled A to I in the figure. Let the distances x, y, z be unknown. Then, if the areas K of the small triangles are equated, we can solve for x, y, z.

From

$$K(AGH) = K(AFE) = K(ACH) = xy$$
$$= K(EIC) = (1-2y-2z+2xz)/4$$

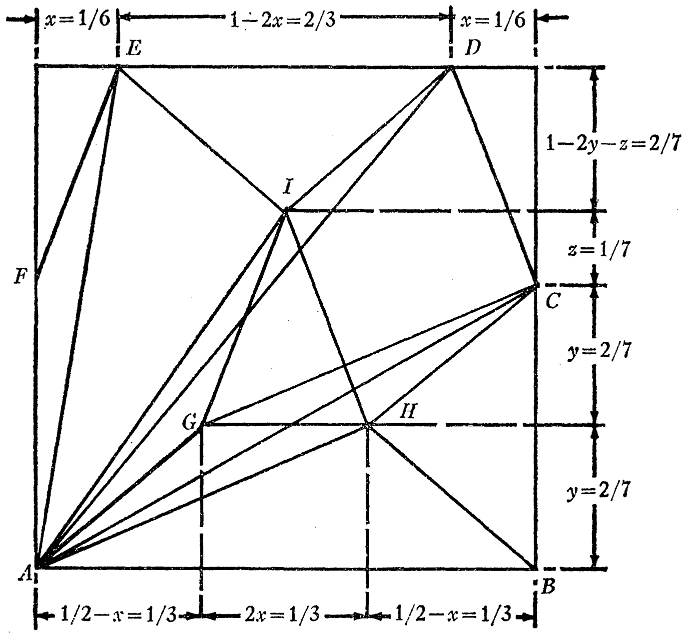


FIG. 1. Nine Points in square
Area of smallest triangle $=1/21=0.04762$.

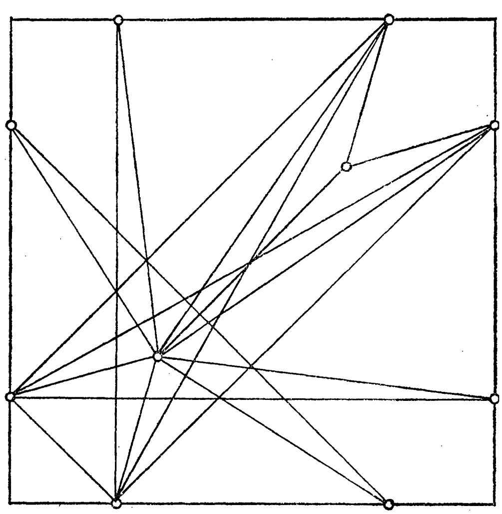


FIG. 2. Ten points in Square
Area of smallest triangle $=K=(3\sqrt{17}-11)/32=0.0428$.

$$= K(EIH) = (y + z - 2x + 2xy)/4$$
$$= K(AGI) = (y + z - 4xy - 2xz)/4,$$

we obtain $x=1/6$, $y=2/7$, $z=1/7$ and $K=1/21=0.04762$.

Other arrangements of the nine points were considered by the proposer, but none yielded a greater least triangle. A rigorous proof of the nonexistence of a better arrangement has not been developed.

A Square Number with Five Fours

786. [January, 1971] *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada.*

Let N be an integer with n digits, in the decenary system, such that the $(n-1)$th, $(n-2)$th, $(n-3)$th, $(n-4)$th, and $(n-5)$th digits are all 4's. What is the smallest N that is a perfect square?

Solution by E. P. Starke, Plainfield, New Jersey.

If the $(n-1)$th digit is 4, the nth must be 1 or 4 or 9. We start with $(50a \pm 7)^2 \equiv 49 \pmod{10^2}$ and find x so that

$$(50x \pm 7)^2 \equiv 449 \pmod{10^3}.$$

The solution of this congruence is routine and leads to

$$(500a \pm 107)^2 \equiv (500a \pm 143)^2 \equiv 449 \pmod{10^3}.$$

Adding a digit at a time we find similarly in succession

$$(5000a \pm 393)^2 \equiv (5000a \pm 857)^2 \equiv 4449 \pmod{10^4},$$
$$(50000a \pm 15393)^2 \equiv (50000a \pm 15857)^2 \equiv 44449 \pmod{10^5},$$
$$(500000a \pm 15857)^2 \equiv (500000a \pm 234707)^2 \equiv 444449 \pmod{10^6}.$$

Thus the smallest number whose square $= \cdots 444449$ is 15857.

Similar analysis produces

$$(500000a \pm 82021)^2 \equiv (500000a \pm 175771)^2 \equiv 444441 \pmod{10^6}$$

and the smallest number of this form is 82021.

Now we have $(500a \pm 38)^2 = 444 \pmod{10^3}$ but, for all a, the next digit (the $(n-4)$th) must be odd.

Note that $(324229)^2 = 105124444441$.

If we counted digits from the other end we could find the smallest: $120185^2 = 14444434225$.

Also solved by Leon Bankoff, Los Angeles, California; Richard L. Breisch, Pennsylvania State University; Mannis Charosh, Brooklyn, New York; Heiko Harborth, Braunschweig, Germany; E. F. Schmeichel and Dave Harris (jointly), College of Wooster, Ohio; Kenneth M. Wilke, Topeka, Kansas; Gregory Wulczyn, Lewisburg, Pennsylvania; and the proposer. Five incorrect solutions were received.

A Match Stick Problem

787. [January, 1971] *Proposed by T. J. Kaczynski, Lombard, Illinois.*

Suppose we have a supply of matches of unit length. Let there be given a square sheet of cardboard, n units on a side. Let the sheet be divided by lines into n^2 little squares. The problem is to place matches on the cardboard in such a way that: a) each match covers a side of one of the little squares, and b) each of the little squares has exactly two of its sides covered by matches. (Matches are not allowed to be placed on the edge of the cardboard.) For what values of n does the problem have a solution?

I. *Solution by Richard A. Gibbs, Hiram Scott College, Nebraska.*

A necessary and sufficient condition that a solution exist is that n be even.

Sufficiency is easy. If $n=2k$, consider the cardboard as consisting of k^2 2×2 squares. Simply place a match on each of the four segments adjacent to the center point of each 2×2 square.

For necessity, assume a solution exists for an $n\times n$ sheet of cardboard. To each unit square correspond the point at its center. Connect two points if their corresponding squares share a match. By the hypotheses, every point will be joined to exactly two others. Therefore, according to a basic result of Graph Theory, the resulting graph will be a collection of disjoint cycles. Each cycle will enclose a polygonal region whose sides are either horizontal or vertical line segments. Consequently, since the length of each segment is an integer, the area of each polygonal region will be an integer. By Pick's theorem (a beautiful result familiar to anyone who has played with a geo-board) the area of the ith polygonal region is

$$A = \tfrac{1}{2}P_i + I_i - 1$$

where there are P_i points on the perimeter and I_i points in the interior of the ith polygonal region. Since each area is an integer, each P_i is even. As each point is on exactly one perimeter, the sum of the P_i is the total number of points, n^2. Hence n is even.

II. *Solution by Richard L. Breisch, Pennsylvania State University.*

A generalization of the stated problem will be demonstrated. Let the cardboard be an $m\times n$ rectangle. The problem of covering the cardboard in the stated manner has a solution if and only if m and $n \geqq 2$, and m and n are not both odd.

An alternative representation of the problem will be used to demonstrate this. Consider the $m\times n$ array of the center points of the little squares. If two edge-adjacent squares have a match on their mutual edge, connect the centers of these squares with a line segment. Since each little square has exactly two of its sides covered by matches, in the alternative representation, there are exactly two line segments from each point in the array. Hence each connected set of line segments forms a polygon, and the $m\times n$ array is covered by a collection of polygons. Each polygon must have an even number of horizontal segments and an even number of vertical segments. Since there are $m\cdot n$ segments, m and n cannot both be odd integers.

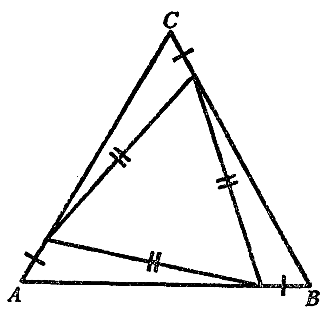
Suppose m is even. Then the $m\times n$ array can be covered with $m/2$ rectangular polygons each of which has dimensions 1 segment by n segments. The arrangement of matches in the original representation is easily derived from this representation.

Also solved by Dan Bean, Dave Harris and E. F. Schmeichel (jointly), College of Wooster, Ohio; Thomas A. Brown, Santa Monica, California; Melvin H. Davis, New York University; Roger Engle and Necdet Ucoluk (jointly), Clarion State College Pennsylvania; Michael Goldberg, Washington, D.C.; M. G. Greening, University of New South Wales, Australia; Heiko Harborth, Braunschweig, Germany; Herbert R. Leifer, Pittsburgh, Pennsylvania; Joseph V. Michalowicz, Catholic University of America; George A. Novacky, Jr., University of Pittsburgh; J. W. Pfaendtner, University of Michigan; Sally Ringland, Shippenville, Pennsylvania; Rina Rubenfeld, New York City Community College; E. P. Starke, Plainfield, New Jersey; and the proposer.

Comment on Problem 574

574. [March and November, 1970] *Proposed by NSF Class at University of California at Berkeley.*

Show that the triangle ABC is equilateral.



Comment by K. R. S. Sastry, Makele, Ethiopia.

Michael Goldberg's solution to the case of "Nested Equilateral Triangles" is so general that we can easily extend it to the case of "Nested Regular Polygons of n Sides." By starting with $A > (n-2)\,180/n$ we arrive at the same contradiction $\theta_1 < \theta_2 < \cdots < \theta_n < \theta_1$.

Comment on T46

T46. [November, 1961] The most common proof of the theorem, "The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides", involves drawing a parallel line and using similar triangles. Devise a shorter proof using areas.

[*Submitted by Robert P. Goldberg*]

Comment by W. C. McDaniel, Southern Illinois University, Carbondale.

In the solution given, use is made of the fact that the perpendicular from C to the opposite side is the altitude of both smaller triangles so that their areas are proportional to the bases, m and n. I note that since D is on the angle bisector the perpendicular segments to the sides of the angle BCA are equal and that these are also the altitudes of the two triangles. Thus the areas are also proportional to the bases, a and b. Consequently, $m/n = a/b$.

Comment on Q443

Q443. [November, 1968] Take the digits of a number expressed in any given base and permute them in any order. Prove that the difference between the two numbers is divisible by a number one less than the base.

[*Submitted by Michael Garrick and Mrs. Jack Lochhead*]

Comment by Charles W. Trigg, San Diego, California.

It is well known and easily proven that in any base b, a number is congruent to the sum of its digits modulo $b-1$. Since the number N and the permutation P have the same digit sum S, then $N\equiv S$ and $P\equiv S \bmod(b-1)$, so $N-P\equiv 0 \bmod(b-1)$.

Comment on Q505

Q505. [January, 1971] Solve the differential equation

$$(x-a)(x-b)y''+2(2x-a-b)y'+2y=0.$$

[*Submitted by Gregory Wulczyn*]

Comment by M. S. Klamkin, Ford Motor Company.

The problem can be easily extended and solved to the differential equation

$$(x-a)(x-b)y''+n(2x-a-b)y'+n(n-1)y=0 \qquad (n=2,3,4,\cdots).$$

Letting $y=D^{n-2}z$, the differential equation can be rewritten as

$$D^n\{z(x-a)(x-b)\}=0.$$

Whence,

$$z=\frac{1}{(x-a)(x-b)}\{A_0+A_1x+\cdots+A_{n-1}x^{n-1}\}$$

$$=B_0+B_1x+\cdots+B_{n-3}x^{n-3}+\frac{A}{x-a}+\frac{B}{x-b}\cdot$$

Finally,

$$y=D^{n-2}\left\{\frac{A}{x-a}+\frac{B}{x-b}\right\}=\frac{A'}{(x-a)^{n-1}}+\frac{B'}{(x-b)^{n-1}}\cdot$$

A similar comment was submitted by Leonard J. Putnick, Siena College, New York.

Comment on Q507

Q507. [January, 1971] For what values of N is $7N+55$ a factor of N^2-71?

[*Submitted by David L. Silverman*]

Comment by Charles W. Trigg, San Diego, California.

An alternate approach involving little computation is:

$$(n^2-71)/(7n+55)=(1/49)(49n^2-3479)/(7n+55)$$
$$=(1/49)[7n-55-2(227)/(7n+55)].$$

Hence $7n+55=\pm(1, 2, 227, \text{ or } 454)$. Only -1 and $+454$ lead to integral values, namely: $n=-8$ and 57.

Comment on Q512

Q512. [January, 1971] Find the solution of

$$x^{(x+1)} + x^x - 1 = 0.$$

[*Submitted by Patricia LaFratta*]

Comment by Benjamin L. Schwartz, McLean, Virginia.

The answer may be correct, but the proof definitely is not. The function $f(x) = x\ln x + \ln(x+1)$ is not monotone for $x > 0$. Consider $f'(x) = 1 + \ln x + 1/(1+x)$. This is obvious negative for small enough, positive x. Interpolation gives $x_0 = 0.154742$ as the only zero of f'. Hence for $x < x_0$, f is decreasing. The published solution is the only one in the open interval $x > x_0$. However, there may be another for $x < x_0$. Letting x approach zero, we find $\lim_{x\to 0} f(x) = 0$ (using l'Hospital's rule, if necessary, to evaluate the indeterminate form $0 \ln 0$). Hence if x is permitted to assume the value 0, with f being interpreted as taking its limiting value, we do indeed have a second solution, viz., $x = 0$.

Comment on Q513

Q513. [March, 1971] Twelve numbers are in arithmetical progression such that $a_k + d = a_{k+1}$. Find the volume of the tetrahedron with vertices (a_1^2, a_2^2, a_3^2), (a_4^2, a_5^2, a_6^2), (a_7^2, a_8^2, a_9^2), $(a_{10}^2, a_{11}^2, a_{12}^2)$

[*Submitted by Charles W. Trigg*]

Comment by Sid Spital, California State College at Hayward.

The answer also follows from the determinant (or scalar triple product) expression for tetrahedron volume:

$$\pm V = \frac{1}{6}\begin{vmatrix} a_4^2 - a_1^2 & a_5^2 - a_2^2 & a_6^2 - a_3^2 \\ a_7^2 - a_1^2 & a_8^2 - a_2^2 & a_9^2 - a_3^2 \\ a_{10}^2 - a_1^2 & a_{11}^2 - a_2^2 & a_{12}^2 - a_3^2 \end{vmatrix} = \frac{(3d)^3}{6}\begin{vmatrix} a_4 + a_1 & a_5 + a_2 & a_6 + a_3 \\ a_7 + a_1 & a_8 + a_2 & a_9 + a_3 \\ a_{10} + a_1 & a_{11} + a_2 & a_{12} + a_3 \end{vmatrix}.$$

Subtracting the second row from the third and then the first row from the second shows that $V = 0$.

ANSWERS

A528. $781 \equiv 7 \pmod 9$ and $965 \equiv 2 \pmod 9$. Thus 753, $A65 \equiv 5 \pmod 9$ and $A = 6$.

A529. Let $x = 1 - t$. The given integral becomes

$$\int_0^1 (1 - x^n)/(1 + x)dx = \int_0^1 1 + x + \cdots + x^{n-1}dx.$$

A530. Since

$$\binom{a-2}{5} = \frac{(a-1)(x+1)(a^3 - 4a)}{120}$$

is an integer and since both primes are not divisible by 2, 3, 4 or 5 we see that 120 must divide a^3-4a.

A531. The answer is no, since such a set could be interpreted as the graph of a continuous contraction of a closed disc on its boundary.

(Quickies on pages 287–288.)

ACKNOWLEDGMENT

The editors wish to thank the following mathematicians for their assistance in refereeing manuscripts submitted to the MATHEMATICS MAGAZINE: G. Angwin, R. A. Askey, B. C. Berndt, J. W. Bishir, R. P. Boas, A. M. Bruckner, P. S. Bullen, E. E. Burniston, I. W. Burr, D. M. Burton, J. B. Carrell, R. E. Chandler, B. Chang, S. R. Clemens, L. J. Cote, H. S. M. Coxeter, K. W. Crain, R. D. Crumley, C. W. Dodge, W. G. Dotson, D. Drasin, M. Drazin, U. Dudley, J. B. Fraleigh, J. M. Gandhi, C. Goffman, M. Golomb, D. H. Gottlieb, J. Hall, Jr., W. J. Harrington, J. E. Huneycutt, A. J. Insel, E. Jacobs, J. H. Jordan, R. B. Killgrove, A. Kömhoff, T. Laetsch, L. Lefevre, C. T. Long, R. E. Lynch, G. R. MacLane, E. A. Maier, M. Marcus, J. A. Marlin, N. H. McCoy, D. S. Moore, W. O. J. Moser, C. J. Neugebauer, M. F. Neuts, P. A. Nickel, L. B. Page, S. S. Page, S. Perlis, J. V. Philipp, C. R. Putnam, R. T. Ramsey, K. V. R. Rao, J. Roberts, N. J. Rose, E. P. Rozycki, J. E. Rubin, S. Samuels, E. Schenkman, R. Schultz, D. E. Scott, M. E. Shanks, T. W. Shilgalis, J. Siegel, L. Spence, E. P. Starke, S. Steinberg, R. A. Struble, P. Swinnerton-Dyer, J. T. Tate, J. K. Thurber, C. W. Trigg, D. F. Ullrich, C. L. Vanden Eynden, J. Vilms, W. A. Webb, S. Weingram, A. Weitsman, V. Weston, R. Westwick, J. V. Whittaker, A. Wilansky, J. W. Yackel, P. R. Young, R. E. Zink.

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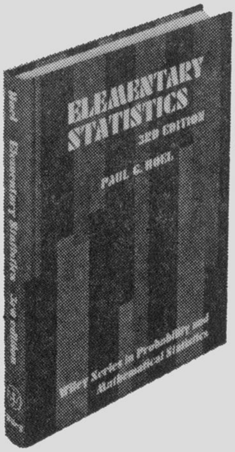

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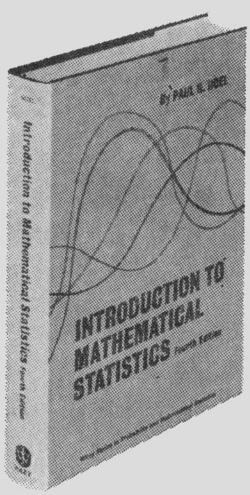

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